How Coherent are Coherent States?

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We review the definition of the canonical coherent state, as proposed by Roy Glauber in 1963, and prove the equivalence between its notable properties. We then generalise the notion of quantum coherence, by first defining the set of incoherent states. Throughout this manuscript, we consider methods of quantifying the coherence of a given quantum state. Upon defining linear functionals and matrix norms, we establish the tools required to construct coherence measures. This leads to the review of the conditions that functionals must satisfy in order to be considered appropriate quantifiers of coherence. By studying three examples and three counter-examples of coherence measures, we analyse the difficulty of constructing such functionals. We then expand the idea of quantifying coherence into infinite-dimensional Hilbert spaces, providing an extra property that finite-dimensional coherence measures must satisfy in order to be suitable quantifiers of coherence in infinite-dimensional systems. We conclude by quantifying the coherence of the canonical coherent state, using the relative entropy of coherence.

1. Introduction

In quantum mechanics, we illustrate the behaviour of particles, such as electrons, using wavefunctions [1]. These wavefunctions mathematically describe the quantum state in which the particle resides. We can describe a system of multiple waves in quantum mechanics as a linear combination of states. For instance, a system of n waves - individually described by a state $|\varphi_i\rangle$ for each $i = 1, \ldots, n$ - is itself a quantum state and can be written as $|\varphi_{\text{sys.}}\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle + \ldots + c_n|\varphi_n\rangle$ where $c_1, c_2, \ldots c_n \in \mathbb{C}$. If we calculate the expectation of an operator $\hat{A}$ in a linear combination of states $|\varphi_{\text{sys.}}\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle$, we can then comment on the coherence of the system.

$$\begin{aligned}\hat{A}_\varphi &= \langle\varphi_{\text{sys.}}|\hat{A}|\varphi_{\text{sys.}}\rangle = |c_1|^2 \langle\varphi_1|\hat{A}|\varphi_1\rangle + |c_2|^2 \langle\varphi_2|\hat{A}|\varphi_2\rangle \\ &\qquad + 2\text{Re}(\bar{c_1}c_2) \langle\varphi_1|\hat{A}|\varphi_2\rangle\end{aligned}$$

If $c_1, c_2 \in \mathbb{C}$, we may write them as $c_1 = r_1 e^{i\theta_1}$ and $c_2 = r_1 e^{i\theta_2}$. Thus, $\bar{c_1}c_2 = r_1 r_2 e^{i(\theta_2 - \theta_1)}$. For this system of two waves to be perfectly coherent, we require the phase difference, $(\theta_2 - \theta_1)$, to be constant. This is intuitive from the classical definition of coherence [2]: multiple waves are known to be perfectly coherent if they have a constant phase difference and equal frequency.

It was only up until very recently that mathematicians and physicists considered quantifying the coherence of a quantum state [3], with 2014 marking the startline for the race to find as many methods of quantification as possible. There are two well-documented functionals satisfying the conditions that have to be met in order to be called coherence measures [4]. We shall discuss these functionals, and their properties, later.

Coherence is a basis-dependent concept. Throughout this paper, we shall be working in the number basis $\{|n\rangle \mid n \in \mathbb{N} \bigcup \{0\}\}$, first defined by Fock in 1932 [5]. This is an orthonormal basis that spans an infinite-dimensional inner product space, known as a Hilbert space [6]. That is to say for any $n, m \in \mathbb{N} \bigcup \{0\}$, we have that $\langle n|m\rangle = \delta_{m,n}$. From the fact that the number basis spans an infinite-dimensional Hilbert space, any state $|\varphi\rangle$ can be written as $|\varphi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$ for all $n \in \mathbb{N} \bigcup \{0\}$. An operator that we shall use throughout this manuscript is the annihilation operator, a. It is defined in the number basis as follows: $a|n\rangle = \sqrt{n}|n-1\rangle$. It is intuitive to use the number basis when considering canonical coherent states because they are defined explicitly in terms of their expansion in the number basis. We shall see such definitions shortly.

In 1926, Erwin Schrödinger attempted to find a "classical" solution to the Quantum Harmonic Oscillator problem [7], similar to Planck's linear oscillator idea [8]. In his landmark paper, Schrödinger defined such a state solution as a "minimum uncertainty" Gaussian wavepacket. Initially, Schrödinger only considered the "minimum uncertainty" states that satisfied the quantum harmonic oscillator problem. 35 years later, Roy Glauber - completely independent of Schrödinger - constructed three general definitions for such "minimum uncertainty" states [9]. Glauber named them "Coherent States".

1.1. **Properties of the Canonical Coherent State.** Roy Glauber gave three properties that a Coherent State must satisfy. In modern literature, these states are referred to as Canonical Coherent States. We will use the same nomenclature here. We can prove all three of his statements, with the detailed method for doing so shown below.

Definition 1.1. The coherent state, $|\alpha\rangle$, is defined to be the normalised eigenstate of the annihilation operator, a:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \qquad \alpha \in \mathbb{C}$$

Thus, the coherent state may be written as $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$

We shall prove that the explicit equation for $|\alpha\rangle$, given in Definition 1.1, is that of a canonical coherent state if and only if $|\alpha\rangle$ is an eigenstate of the annihilation operator.

Proof. [$\Rightarrow$] We first start off by proving the fact that the canonical coherent state is an eigenstate of the annihilation operator, using the explicit definition of $|\alpha\rangle$ stated above. Recalling the action of the annihilation operator on the number basis, $a|n\rangle = \sqrt{n}|n-1\rangle$, the eigenstate property becomes straightforward to prove.

$$\begin{aligned} a|\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \left[\frac{\alpha^n}{\sqrt{n!}}\right] \sqrt{n}|n-1\rangle = \alpha e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle \\ &= \alpha e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{(m)!}}|m\rangle \qquad \text{Having relabelled } (n-1) \to m \\ &= \alpha|\alpha\rangle \end{aligned}$$

Hence, we have just shown that the canonical coherent state is indeed an eigenstate of the annihilation operator. ■

Proof. [⇐] We must now prove that the definition of any normalised eigenstate of the annihilation operator, say $|\varphi\rangle$, matches that of the canonical coherent state defined above. We start off by recalling the fact that any state $|\varphi\rangle$ can be written as $|\varphi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, where $c_n \in \mathbb{C}$. We may apply the annihilation operator to this state: $a|\varphi\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n} |n-1\rangle$. At the beginning of this proof, we assumed that $|\varphi\rangle$ is an eigenstate of the annihilation operator. That is to say, for some eigenvalue $\alpha \in \mathbb{C}$, we have that $a|\varphi\rangle = \alpha|\varphi\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$. For these two states to be equal, we must have that their components are equal too. Upon setting these two equations for $a|\varphi\rangle$ to be equal, we get the following relation.

$$\sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$$

Knowing that the number basis is an orthonormal basis of our Hilbert Space, we may think to apply a bra $\langle m|$ to this relation. We then achieve the following, more intuitive, condition that the coefficients must satisfy in order for the two expressions for $a|\varphi\rangle$ to be equal.

$$\begin{aligned}\sum_{n=1}^{\infty} c_n \sqrt{n} \langle m|n-1\rangle &\overset{!}{=} \sum_{n=0}^{\infty} \alpha c_n \langle m|n\rangle \\ \Rightarrow \quad c_{m+1}\sqrt{m+1} &= \alpha c_m\end{aligned}$$

We now have an equation to solve in order to find the coefficients of $|\varphi\rangle$. Starting from $c_0 = C$, we can proceed by considering the first few terms of the iteration.

$$\begin{aligned} c_1 &= \frac{\alpha}{\sqrt{1}} c_0 = \frac{\alpha}{\sqrt{1}} C \\ c_2 &= \frac{\alpha}{\sqrt{2}} c_1 = \frac{\alpha^2}{\sqrt{2\cdot 1}} C \\ c_3 &= \frac{\alpha}{\sqrt{3}} c_2 = \frac{\alpha^3}{\sqrt{3\cdot 2\cdot 1}} C \\ c_4 &= \frac{\alpha}{\sqrt{4}} c_3 = \frac{\alpha^4}{\sqrt{4\cdot 3\cdot 2\cdot 1}} C \end{aligned}$$

By induction, we see that $c_n = \frac{\alpha^n}{\sqrt{n!}} C$. Hence, $|\varphi\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C|n\rangle$. In order to evaluate C, we must consider the fact that $|\varphi\rangle$ is normalised. That is, we require the following to hold.

$$\begin{aligned} 1 \overset{!}{=} \langle\varphi|\varphi\rangle &= \sum_{n,m=0}^{\infty} \frac{\bar{\alpha}^n \alpha^m}{\sqrt{m!}\cdot\sqrt{n!}} C^2 \langle m|n\rangle = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} C^2 \\ &= e^{|\alpha|^2} C^2 \end{aligned}$$

Therefore, we have that $C = e^{-\frac{|\alpha|^2}{2}}$ and hence $|\varphi\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$. Thus, $|\varphi\rangle$ gives the definition of a canonical coherent state, from the fact that it is a normalised eigenstate of the annihilation operator. ■

Theorem 1.2. The coherent state $|\alpha\rangle$ can be obtained by applying the displacement operator, $D(\alpha)$, to the vacuum state of the quantum harmonic oscillator, $|0\rangle$.

$$|\alpha\rangle = D(\alpha)|0\rangle, \text{ where } D(\alpha) = e^{\alpha a^\dagger - \bar{\alpha} a}$$

It is also worth noting that the α present here is the same α defined in Definition 1.1, α is an eigenvalue of the annihilation operator.

Proof. We may rewrite the displacement operator in a different way, using a variant of the Baker-Campbell-Hausdorf formula. The formula states that, for not necessarily commuting Hilbert Space operators X and Y, $e^X e^Y e^{-\frac{1}{2}[X,Y]} = e^{X+Y}$. By first noticing that $[a^\dagger, a] = -\mathbb{1}$, we can apply this formula to the displacement operator defined by $D(\alpha) = e^{\alpha a^\dagger - \bar{\alpha} a}$.

Using $X = \alpha a^\dagger$ and $Y = -\bar{\alpha} a$ in the Baker-Campbell-Hausdorf formula, we get the following result.

$$D(\alpha) = e^{\alpha a^\dagger - \bar{\alpha} a} = e^{\alpha a^\dagger} e^{-\bar{\alpha} a} e^{-\frac{\left[\alpha a^\dagger, -\bar{\alpha} a\right]}{2}} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\bar{\alpha} a}$$

This is a result that was explained in detail by Mandel [10]. We may now use the Taylor Series of the exponential function, centred at the origin, to rewrite the displacement operator further.

$$D(\alpha) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha a^\dagger\right)^n}{n!} \sum_{m=0}^{\infty} \frac{(-\bar{\alpha} a)^m}{m!}$$

In order to see the action of the displacement operator on $|0\rangle$, we must first see how $\sum_{m=0}^{\infty} \frac{(-\bar{\alpha} a)^m}{m!}$ operates on the vacuum state.

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{(-\bar{\alpha} a)^m}{m!}|0\rangle &= \frac{(-\bar{\alpha} a)^0}{0!}|0\rangle + \frac{(-\bar{\alpha} a)^1}{1!}|0\rangle + \frac{(-\bar{\alpha} a)^2}{2!}|0\rangle + \frac{(-\bar{\alpha} a)^3}{3!}|0\rangle + \dots \\ &= |0\rangle - \bar{\alpha} a|0\rangle + \frac{\bar{\alpha}^2 a^2}{2}|0\rangle - \frac{\bar{\alpha}^3 a^3}{6}|0\rangle + \dots \\ &= |0\rangle \text{ because } a|0\rangle = 0\end{aligned}$$

This shows us that the first summation term of the displacement operator behaves like the identity operator on $|0\rangle$. This simply means that we can disregard this term of the displacement operator when considering the action of $D(\alpha)$ on the vacuum state. We can now discuss the action of the remaining terms of the displacement operator on the vacuum state.

$$\begin{aligned}e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha a^\dagger\right)^n}{n!}|0\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left(a^\dagger\right)^n |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sqrt{n!}|n\rangle \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle\end{aligned}$$

Collecting the action of both summation terms of the displacement operator, we see that indeed the canonical coherent state is produced as a result of the displacement operator acting on the vacuum state. ■

The next statement involves the position and momentum operators, denoted by $\hat{q}$ and $\hat{p}$ respectively. These operators act on a wavefunction, $\psi(x,t)$, in the following way.

$$\hat{q}\psi(x,t) = x\psi(x,t) \qquad \hat{p}\psi(x,t) = -i\hbar\frac{\partial\psi(x,t)}{\partial x}$$

This is where $\hbar$ is known as Planck's constant. These operators can be written in terms of annihilation and creation operators, using the Stone-Von Neumann uniqueness theorem [11] and the fact that $[a, a^\dagger] = aa^\dagger - a^\dagger a = \mathbb{1}$. The next statement is given below.

Theorem 1.3. The coherent state $|\alpha\rangle$ is a quantum state with a minimum uncertainty property. That is to say,

$$(\Delta_\alpha \hat{q})^2(\Delta_\alpha \hat{p})^2 = \left(\frac{1}{2}\right)^2$$

This becomes easier to see if we define the coordinate and momentum operators $(\hat{q}, \hat{p})$ in the following way.

$$\hat{q} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right) \qquad \hat{p} = \frac{1}{i\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right)$$

Proof. We shall prove that the canonical coherent state is indeed a quantum state with a minimum uncertainty property. From Definition 1.1, we have that $a|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha \in \mathbb{C}$ and $\langle\alpha|\alpha\rangle = 1$. This implies that both $\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$ and $\langle\alpha|a^\dagger = \langle\alpha|\bar{\alpha}$ are true. Thus, we can use the linearity of these quantum operations to assert that the following results hold.

$$\langle\alpha|(a + a^\dagger)|\alpha\rangle = (\alpha + \bar{\alpha})$$
$$\langle\alpha|(a - a^\dagger)|\alpha\rangle = (\alpha - \bar{\alpha})$$
$$\langle\alpha|(a + a^\dagger)(a + a^\dagger)|\alpha\rangle = (\alpha + \bar{\alpha})^2 + 1$$
$$\langle\alpha|(a - a^\dagger)(a - a^\dagger)|\alpha\rangle = (\alpha - \bar{\alpha})^2 - 1$$

Using these identities, we can compute the variance of the coordinate and momentum operators in the canonical coherent state. Referring to the definitions of $\hat{p}$ and $\hat{q}$ in terms of creation and annihilation operators, we can compute the following results.

$$(\Delta_\alpha \hat{p})^2 = \left\langle \hat{p}^2 \right\rangle_\alpha - (\langle\hat{p}\rangle_\alpha)^2 = \frac{1}{2}\langle\alpha|(a + a^\dagger)^2|\alpha\rangle - \frac{1}{2}\left(\langle\alpha|(a + a^\dagger)|\alpha\rangle\right)^2$$
$$= \frac{1}{2}((a + a^\dagger)^2 + 1 - (a + a^\dagger)^2) = \frac{1}{2}$$

We can then follow a similar argument for the variance of the position operator, $\hat{q}$.

$$(\Delta_\alpha \hat{q})^2 = \left\langle \hat{q}^2 \right\rangle_\alpha - (\langle \hat{q} \rangle_\alpha)^2 = -\frac{1}{2}\langle\alpha|(a-a^\dagger)^2|\alpha\rangle + \frac{1}{2}\left(\langle\alpha|(a-a^\dagger)|\alpha\rangle\right)^2$$
$$= \frac{1}{2}(-(a-a^\dagger)^2 + 1 + (a-a^\dagger)^2) = \frac{1}{2}$$

We can multiply these two results together to prove that $|\alpha\rangle$ is indeed a minimum uncertainty state. That is to say that it saturates the equality in Heisenberg's uncertainty relation: $(\Delta_\alpha \hat{q})^2(\Delta_\alpha \hat{p})^2 = \left(\frac{1}{2}\right)^2$. ■

Glauber's definition of the canonical coherent state describes only a small subset of quantum states that are said to be coherent. The coherence of a quantum state may take a range of values and can indeed be measurable. It is easier to define a set of states that are not coherent rather than the set of states that possess some coherence. In the next section, we shall define such incoherent states and discuss some necessary background mathematics. This will set the table for analysis to begin on coherence measures.

2. Incoherence, Functionals and Matrix Norms

In this section, we shall construct the framework necessary to understand how to quantify coherence. We first begin by defining the set of states that are not coherent, known as incoherent states.

2.1. **Incoherent States.** As mentioned previously, coherence is a basis-dependent concept. That is to say, if a state $|\varphi\rangle$ can be represented in multiple distinct bases spanning the same Hilbert space, then the coherence of $|\varphi\rangle$ may not be equal to the coherence of the same state in different basis representations. This will become clear when we define incoherent states. Using the line of thought given by Streltsov [12], we state that incoherence must be defined in a chosen basis: We define what a d-dimensional incoherent state is with respect to the number basis.

Definition 2.1. An d-dimensional incoherent state, $\hat{\delta}$, is defined to be a diagonal density matrix with respect to the chosen basis of the d-dimensional Hilbert space, $\mathcal{H}$. That is to say, an incoherent state defined in the number basis is given by the following equation.

$$\hat{\delta} = \sum_{i=0}^{d-1} \delta_i |i\rangle\langle i|$$

The δ_i satisfy both $0 \leqslant \delta_i \leqslant 1$ and $\sum_{i=0}^{d-1} \delta_i = 1$. We call all density matrices that are diagonal in this space incoherent, i.e., in the set of incoherent states, I.

From Definition 2.1, we can claim that a state, $\hat{\rho}$, is coherent if and only if $\hat{\rho} \notin I$. Speaking more abstractly, if the state $\hat{\rho}$ has at least one non-zero off-diagonal element, then we can say that $\hat{\rho}$ possesses some coherence. We have just seen that quantum states can be classified into either coherent states or incoherent states. The same is true for quantum operations.

2.2. **Quantum Operations.** The notion of quantum operations is far from specific, describing a subset of mathematical transformations that a quantum system can undergo. Throughout this manuscript, we will be only considering those operations that act on density operators of quantum states. Karl Kraus concretely defined such operations in terms of matrices, known as Kraus Operators [13].

Theorem 2.2 (Kraus's Theorem)**.** Let $\mathcal{H}$ and $\mathcal{H}'$ be d-dimensional Hilbert Spaces. Let Φ be a quantum operation mapping the density matrices acting on $\mathcal{H}$ to those acting on $\mathcal{H}'$. Then there exists matrices $\{K_n\}_{1\leqslant n\leqslant d}$, and their corresponding Hermitian conjugates $\{K_n^\dagger\}_{1\leqslant n\leqslant d}$, such that Φ can be written in the following way.

$$\Phi(\hat{\rho}) = \sum_n K_n \hat{\rho} K_n^\dagger \text{ and } \sum_n K_n^\dagger K_n \leqslant \mathbb{1}$$

The quantum operation Φ is known as a trace-preserving operation if and only if $\sum_n K_n^\dagger K_n = \mathbb{1}$

A variant of this theorem was proved nicely by Nielsen and Chuang [14]. Furthermore, we will only be implementing quantum operations that are trace-preserving: This property is explicitly defined by the mathematical condition imposed on the Kraus Operators. These trace-preserving quantum operations are known as channels and will be the main focus of this section. It is important to note that some authors use the term Completely Positive Trace-Preserving Map instead of Quantum Channel, as discussed by Weedbrook [15]. Just like quantum states, we can classify quantum operations into coherent operations and incoherent operations.

Definition 2.3. Let $\hat{\delta} \in I$ and let Φ be a quantum channel. The quantum channel Φ is said to be an incoherent quantum channel if and only if $\Phi(\hat{\delta}) \in I$ for all $\hat{\delta} \in I$. In terms of its Kraus decomposition, this can be informally written as $\sum_n K_n I K_n^\dagger \subset I$

To illustrate this further, here is an example of a quantum operation that we shall use extensively throughout this manuscript. It is described much more comprehensively by Imre [16].

Definition 2.4. The Dephasing map is a quantum operation, Φ, that acts on a quantum state, $\hat{\rho}$, in the following way.

$$\hat{\rho} \;\mapsto\; \Phi(\hat{\rho}) = \sum_n |n\rangle\langle n|\hat{\rho}|n\rangle\langle n|$$

Colloquially speaking, we say that this mapping diagonalises any density matrix, $\hat{\rho}$.

We will soon see how important incoherent quantum channels are when investigating the worthiness of coherence measures. For now, we shall state another mathematical concept that is essential in the process of constructing coherence measures.

2.3. **Defining Functionals and Matrix Norms.** In order to quantify coherence, we require a mapping that takes the set of density matrices in a Hilbert Space to the set of real numbers. This is known as a functional. We can define this formally [17].

Definition 2.5. A real, linear functional, f, is a mapping from a vector space X to the real numbers, $\mathbb{R}$. That is to say,

$$X \mapsto f[X] \subseteq \mathbb{R}$$

We will only consider the functionals that operate on the Hilbert Space.

In pursuance of constructing a functional that maps from the set of density matrices to $\mathbb{R}$, we may consider the following well-known functional.

Definition 2.6. A Matrix Norm is a function that assigns a strictly positive length to each matrix in the vector space $\mathbb{C}^{m\times n}$.
It is a mapping $\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$ that satisfies the following properties.

For all matrices A, $B \in \mathbb{C}^{m\times n}$ and for all scalars $\alpha \in \mathbb{C}$:

- $\|A\| \geqslant 0$
- $\|A\| = 0$ if and only if $A_{ij} = 0$ for all $1 \leqslant i, j \leqslant m, n$
- $\|\alpha A\| = |\alpha| \, \|A\|$
- $\|A + B\| \leqslant \|A\| + \|B\|$

Matrix norms are a key ingredient in the construction of coherence measures. A more comprehensive definition of the matrix norm was given by Meyer [18]. We shall discuss examples of matrix norms below.

Naively, the first thing that comes to mind when thinking of matrix norms are the entry-wise matrix norms. These are norms that treat $A \in K^{m\times n}$ as a vector of size mn. One of the most general entry-wise norms is defined in the following way, with a more comprehensive definition given by Ding et. al. [19].

Definition 2.7. For numbers $p, q \in \mathbb{N}$, we define the $L_{p,q}$ norm of a matrix $A \in K^{m\times n}$ using the subsequent formula.

$$\|A\|_{p,q} = \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{ij}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

There is a specific case of the $L_{p,q}$ norm that we shall use later in this manuscript. It is stated as an explicit example below.

Example 2.8. The well-known Hilbert-Schmidt norm, also known as the Frobenius norm, of a matrix $A \in K^{m\times n}$ can be defined as the $L_{p,q}$ norm when $p = q = 2$. That is,

$$\|A\|_{HS} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\text{trace}(A^\dagger A)}$$

A more comprehensive discussion of the Hilbert-Schmidt norm is given by Reed and Simon [20]. Now we must discuss a more important type of norm in the theory of coherence measures: the Schatten p-norm [21].

Definition 2.9. For a number $p \in \mathbb{N}$, we define the Schatten p-norm of a matrix $A \in K^{m\times n}$ using the following equation.

$$\|A\|_p = \left(\sum_{n \geqslant 1} s_n^p(A) \right)^{\frac{1}{p}}$$

The scalars $s_1(A) \geqslant s_2(A) \geqslant \ldots \geqslant s_n(A) \geqslant \ldots \geqslant 0$ are the singular values of A and the eigenvalues of the Hermitian operator $|A| = \sqrt{(A^\dagger A)}$.

The Schatten p-norm can also be written as $\|A\|_p^p = \text{tr}(|A|^p)$

As we have done with the $L_{p,q}$ entry-wise norm, we shall provide a specific example of the Schatten p-norm and study its properties. The most trivial, and versatile, form of the Schatten p-norm is the l_1 norm: we define it below.

Example 2.10. The l_1 norm of a matrix $A \in K^{m\times n}$ is defined as the Schatten p-norm of the matrix A when $p = 1$.

$$\|A\|_1 = \sum_{n \geqslant 1} s_n(A)$$

Also known as the trace norm; it may be written as $\|A\|_1 = \text{tr}(|A|)$

As an aside, recall that any function $f : \mathbb{R} \to \mathbb{R}$ applied to a Hermitian operator A is as follows [22].

$$f(A) = \sum_n f(a_n)|n\rangle\langle n|$$

This is where $\sum_n a_n|n\rangle\langle n|$ is the spectral decomposition of A. We say that $|n\rangle\langle n| \equiv P_n$ are the spectral projections of A. We also know that the normal matrix A can be written as $A = UMU^\dagger$ [23]. With these facts, it is trivial to show that the trace norm of M is the sum of the absolute values of its eigenvalues, i.e. $\|M\|_1 = \sum_n |\lambda_n|$. It is beneficial to note that we can also write the spectral decomposition of M as $M = \sum_n a_n|n\rangle\langle n|$.

The trace norm is indeed a norm because the subsequent properties can be proven to be satisfied: homogeneity, positive definiteness, and the triangle inequality [24]. One coherence measure I shall discuss in this manuscript will involve the l_1 norm of the half-distance between two density matrices, $\hat{\rho}$ and $\hat{\sigma}$. This is known as the trace distance [25].

There is a rather inclusive set that we require all states and density matrices to be an element of in finite-dimensional Hilbert spaces. Every element of this set is well-defined under an l_1 norm. Let us explain this further and discuss the importance of this set in our line of investigation.

2.4. **The Importance of Trace Class.** We shall state a property that all density matrices must exhibit, when considering them in finite-dimensional Hilbert spaces, and the significance of it. The following definition gives the condition for which bounded linear operators [26] must satisfy in order to be considered trace class.

Definition 2.11. A bounded linear operator A over a Hilbert space, $\mathcal{H}$, is said to be in the trace class, if the following inequality holds.

$$\|A\|_1 = \text{tr}(|A|) < \infty$$

This can also be written in terms of the inner product and any orthonormal basis $\{e_k\}_{k\in\mathbb{N}\bigcup\{0\}}$.

$$\sum_k \left\langle \sqrt{A^\dagger A}\ e_k, e_k \right\rangle < \infty$$

Definition 2.11 follows from the definition of the trace of an operator, $\text{tr}(A) = \sum_k \langle A\ e_k, e_k\rangle$, as measured in an orthonormal basis, $\{e_k\}$. When the Hilbert space, $\mathcal{H}$, is finite-dimensional, every bounded operator defined on it is trace class. This is evident from the definition: all finite sums are convergent.

It begins to get interesting when we consider bounded linear operators on infinite-dimensional Hilbert spaces. The trace of a linear operator in infinite dimensions is not always defined: Hence, not all operators are trace class in infinite dimensions. When the summation proposed in Definition 2.11 is convergent, we know that A is a trace class operator in infinite dimensions. In order to illustrate this concept, we shall provide an example.

Example 2.12. The dephasing map, Φ, is indeed trace preserving. This means that, if $\hat{\rho}$ is trace class, then $\Phi(\hat{\rho})$ is also trace class. Recall that the dephasing map is defined in the following way.

$$\hat{\rho} \;\mapsto\; \Phi(\hat{\rho}) = \sum_n |n\rangle\langle n|\hat{\rho}|n\rangle\langle n|$$

Since the trace is invariant under a change of basis, we may pick an orthonormal basis we are familiar with: the number basis. The sum we have to test the convergence of takes the following form.

$$\begin{aligned}
\mathrm{tr}(|\Phi(\hat{\rho})|) &= \sum_{n=0}^{\infty} \left\langle \sqrt{\sum_i |i\rangle\langle i|\hat{\rho}^\dagger|i\rangle\langle i| \cdot \sum_j |j\rangle\langle j|\hat{\rho}|j\rangle\langle j|}\; n, n \right\rangle \\
&= \sum_{n=0}^{\infty} \left\langle \sqrt{\sum_i |i\rangle\langle i|\hat{\rho}^\dagger\hat{\rho}|i\rangle\langle i|}\; n, n \right\rangle \\
&= \sum_{n=0}^{\infty} \langle n| \sum_i \left(|i\rangle\langle i|\sqrt{\hat{\rho}^\dagger\hat{\rho}}|i\rangle\langle i| \right) |n\rangle \\
&= \sum_{n=0}^{\infty} \langle n|\sqrt{\hat{\rho}^\dagger\hat{\rho}}|n\rangle = \mathrm{tr}(|\hat{\rho}|) = 1
\end{aligned}$$

Hence, we have just shown that, provided $\hat{\rho}$ is a density matrix representing a quantum state, the dephasing map is trace preserving.

We shall use the dephasing map when constructing coherence measures, taking note of its trace preserving property. We can now discuss the rules that functionals need to satisfy to be called coherence measures.

3. Tenets for Measuring Coherence

We may now lay out the four properties that a coherence measure must exhibit. These properties were first described this way by Baumgratz et. al. [4]. The first characteristic that a coherence measure must possess involves the incoherent state, $\hat{\delta}$.

(C1): A coherence measure, $C(\hat{\rho})$, must vanish upon the insertion of an incoherent state, $\hat{\delta}$. That is to say, for all $\hat{\delta} \in I$,

$$C(\hat{\delta}) = 0$$

This is a rather trivial property, though an important one to note. We require that any coherence measure will return the value of zero upon the measurement of an incoherent state. The next property declares that a coherence measure, $C(\hat{\rho})$, does not increase under incoherent completely positive trace-preserving operations, $\Phi_{\mathrm{ICPTP}}(\hat{\rho})$.

(C2a): A coherence measure, $C(\hat{\rho})$, must be non-increasing under incoherent completely positive trace-preserving operations. This means for all $\Phi_{\mathrm{ICPTP}}(\hat{\rho})$,

$$C(\hat{\rho}) \geqslant C(\Phi_{\mathrm{ICPTP}}(\hat{\rho}))$$

We can write this property in more familair terms, using Kraus operators. As we mentioned previously, incoherent quantum operations may be written as $\Phi(\hat{\rho}) = \sum_n K_n \hat{\rho} K_n^\dagger$, where the Kraus operators satisfy $K_n I K_n^\dagger \subset I$ and are all of the same dimension ($d_{\mathrm{out}} \times d_{\mathrm{in}}$). The trace-preserving property is being encompassed by the fact that $\sum_n K_n^\dagger K_n = \mathbb{1}$. One thing to add is that an operation is completely positive if the Kraus operators defining it are positive definite. For more on this idea, see the work of Bhatia [27]. Therefore, if $\Phi(\hat{\rho}) = \sum_n K_n \hat{\rho} K_n^\dagger$ then (C2a) may be written in the following form.

$$C(\hat{\rho}) \geqslant C\left(\sum_n K_n \hat{\rho} K_n^\dagger\right)$$

This formulation of the ICPTP map is not the general form of a quantum operation, however. This is because the ICPTP map disregards measurement outcomes, meaning that the ICPTP map does not in general retain any information about measurement outcomes. Thus, a more general definition of an incoherent operation is required: one that retains the information regarding measurement outcomes.

An incoherent quantum operation that retains the information regarding measurement outcomes is known as an incoherent measuring operation. Such a definition was proposed by Zhao and Yu [28]. We define such operations again by Kraus operators satisfying both $K_m I K_m^\dagger \subset I$ and $\sum_m K_m^\dagger K_m = \mathbb{1}$. However, the Kraus operators describing the operations are of different dimensions given by $d_m \times d_{\mathrm{in}}$ for each m. Considering the outcomes of the measurement, we say that the state relating to the outcome m is given by

$$\hat{\rho}_m = \frac{K_m \hat{\rho} K_m^\dagger}{p_m}$$

and arises with probability $p_m = \mathrm{tr}[K_m \hat{\rho} K_m^\dagger]$ such that $\sum_m p_m = 1$. This tells us that we can select certain measurement outcomes of the operation: all of the information resulting from the outcome m is encoded by ρ_m and p_m. Thus, we require a stronger property than (C2a) that states that $C(\hat{\rho})$ is non-increasing independent of which measurement outcome we select.

(C2b): A coherence measure, $C(\hat{\rho})$, must be non-increasing under incoherent measuring operations. We can take an average of coherence over all measurement outcomes n, with probabilities p_n, in the following way.

$$C_{\text{ave.}}(\hat{\rho}) = \sum_n p_n C(\hat{\rho}_n)$$

This average of coherence, $C_{\text{ave.}}(\hat{\rho})$, must not be greater than the coherence of the complete state, $\hat{\rho}$. That is to say,

$$C(\hat{\rho}) \geqslant \sum_n p_n C(\hat{\rho}_n)$$

This is a more general form of (C2a) as it allows us to select measurement outcomes of the operation. We will see why we regard (C2b) in higher priority than (C2a) later in this section.

In order to understand the fourth property that a coherence measure must satisfy, we must first discuss what it means for a functional to be convex [29].

Definition 3.1. A real, linear functional f, defined on a convex subset of a vector space X, is convex if and only if, $\forall\ u, v \in X$ and $0 \leqslant \alpha \leqslant 1$, the following inequality holds.

$$f((1-\alpha)u + \alpha v) \leqslant (1-\alpha)f(u) + \alpha f(v)$$

A convex subset of a vector space X is one in which every element of the subset can be connected by a straight line that lies in the subset itself.

The fourth condition that coherence measures must satisfy involves the theory of mixing quantum states. We take a set of density matrices, $\{\hat{\rho}_n\}$, and arrange them into a convex combination, using probabilities p_n as coefficients such that $\sum_n p_n = 1$. We call this a statistical ensemble of quantum states. From the definition of coherence, it would prove intuitive that the statistical ensemble is more incoherent. In a statistical ensemble there are multiple wavefunctions collectively describing the system, therefore, loosely speaking, there is a smaller chance that all of these wavefunctions describing the system have a constant phase relation. Therefore, we require that any coherence quantifier measures this potential decrease in coherence of a system. We can encompass this with our fourth and final condition.

(C3): A coherence measure, $C(\hat{\rho})$, must be non-increasing under mixing of quantum states. This means that, for any set of states $\{\hat{\rho}_n\}$, as described previously, with corresponding probabilities $p_n \geqslant 0$ satisfying $\sum_n p_n = 1$,

$$C(\hat{\rho}) \geqslant \sum_n p_n C(\hat{\rho}_n)$$

This is an essential property that possible candidates need to satisfy in order to be called coherence measures. The importance of convexity, in the context of coherence measures, was discussed in detail by Streltsov et. al. [30]. The importance of all these properties are not equal: It suffices to prove (C1), (C2b) and (C3) to show that a functional is a coherence measure. This argument can be seen below:

$$C(\Phi_{\text{ICPTP}}(\hat{\rho})) = C\left(\sum_n p_n \hat{\rho}_n\right) \overset{\text{(C3)}}{\leqslant} p_n C\left(\sum_n \hat{\rho}_n\right) \overset{\text{(C2b)}}{\leqslant} C\left(\hat{\rho}\right)$$

Hence, we have that the properties (C3) and (C2b) collectively imply (C2a). Therefore, when we begin to discuss candidates for coherence measures, it is sufficient to prove that properties (C1), (C2b) and (C3) are satisfied by candidate functionals.

4. Candidates for Coherence Measures

In this section, I will discuss two widely postulated candidates for coherence measures: The relative entropy [31] and l_1 norm of coherence [32]. I will then provide three original candidates for measuring coherence, analysing the methodology behind constructing such measures. I hope that this section provides some insight into the difficulty of developing coherence measures.

4.1. **Distinguishability Between a State and its Diagonal.** The first coherence measure we will discuss is the relative entropy of coherence. This measure was engineered from the Kullback-Leibler divergence [33]: A measure of how one probability distribution differs from another. For density matrices $\hat{\rho}$ and $\hat{\sigma}$, the Kullback-Liebler divergence of $\hat{\rho}$ and $\hat{\sigma}$ is given by

$$D_{\text{KL}}(P||Q) = \text{tr}\left(\hat{\rho}\log\hat{\rho} - \hat{\rho}\log\hat{\sigma}\right)$$

This looks like a distance measure: One can think of it as measuring the distinguishability of two quantum states. Hence, this could make a suitable measure of coherence, measuring the distinguishability of a state, $\hat{\rho}$, to the closest incoherent state, $\hat{\delta}$.

Definition 4.1. Given a quantum state $\hat{\rho}$ and an incoherent state $\hat{\delta}$, the Relative Entropy of Coherence is defined in the following way.

$$C_{\text{Rel Ent.}}(\hat{\rho}) = \min_{\hat{\delta}\in I}\ \text{tr}\left(\hat{\rho}\log\hat{\rho} - \hat{\rho}\log\hat{\delta}\right)$$

Though the choice of $\hat{\delta} \in I$ may seem arbitrary, Definition 4.1 merely states that we choose the closest incoherent state to $\hat{\rho}$. As we have mentioned previously, it suffices to prove (C1), (C3) and (C2b). Often, it is the latter property that is difficult to prove. We shall prove the conditions in that order, leaving (C2b) for last. The first condition, (C1), is generally the easiest one to prove: Let $\hat{\rho} = \hat{\delta} \in I$ and observe whether the functional vanishes. The following argument can be made for the satisfaction of (C1) by Relative Entropy of Coherence.

(C1): Let $\hat{\rho} = \hat{\delta} \in I$, then the relative entropy of $\hat{\delta}$ and $\hat{\delta}$ gives the following result.

$$\begin{aligned} C_{\text{Rel Ent.}}(\hat{\delta}) &= \min_{\hat{\delta}\in I} \ \operatorname{tr}\left(\hat{\delta}\log\hat{\delta} - \hat{\delta}\log\hat{\delta}\right) \\ &= \min_{\hat{\delta}\in I} \ \operatorname{tr}(0) = 0 \end{aligned}$$

This proves to be intuitive: If both states are equal then they are indistinguishable. Therefore, a returned value of zero makes sense mathematically.

We can now discuss the fourth tenet: (C3). This states that we require the functional to be convex. In order to prove this concisely, we shall consider the satisfaction of (C3) for $n \in \{0, 1\}$.

(C3): We wish to prove that the relative entropy is convex using only four arguments: $\hat{\rho}_0$, $\hat{\rho}_1$, $\hat{\delta}_0$ and $\hat{\delta}_1$, with probabilities p_0 and p_1. We then proceed in the following way.

$$\begin{aligned} &C_{\text{Rel Ent.}}(p_0\hat{\rho}_0 + p_1\hat{\rho}_1) = \\ &= \ \operatorname{tr}\left([p_0\hat{\rho}_0 + p_1\hat{\rho}_1]\log(p_0\hat{\rho}_0 + p_1\hat{\rho}_1) - [p_0\hat{\rho}_0 + p_1\hat{\rho}_1]\log\left(\hat{\delta}_0 + \hat{\delta}_1\right)\right) \\ &\leqslant \ p_0\operatorname{tr}\left(\hat{\rho}_0\log\hat{\rho}_0 - \hat{\rho}_0\log\hat{\delta}_0\right) + p_1\operatorname{tr}\left(\hat{\rho}_1\log\hat{\rho}_1 - \hat{\rho}_1\log\hat{\delta}_1\right) \\ &= \ p_0 \cdot C_{\text{Rel Ent.}}(\hat{\rho}_0) + p_1 \cdot C_{\text{Rel Ent.}}(\hat{\rho}_1) \end{aligned}$$

It can be shown that this proof extends to a general mixture of the system. Hence, we have just shown that the relative entropy satisfies (C3). We must now take a diversion from our investigation, in order to understand the property of (C2b) and how we go about proving its satisfaction. It is difficult to prove this inequality directly. It is far easier to prove smaller properties, as part of a larger argument, to prove that a distance measure can satisfy (C2b). That much larger argument, which uses the relationship between coherence and entanglement measures [34], is explained in detail below. We have to discuss (C2a) first: Then we can state a corollary to bridge the gap toward the examination of (C2b). It is trivial to see that (C2a) boils down to proving the following inequality for any distance measure, $D(\hat{\rho}||\hat{\delta})$.

$$D(\hat{\rho}||\hat{\delta}) \geqslant D(\sum_n K_n\hat{\rho}K_n^\dagger || \sum_n K_n\hat{\delta}K_n^\dagger)$$

This leads to:

Theorem 4.2. For any, not necessarily incoherent, CPTP map Φ_{CPTP}, given by $\Phi_{\text{CPTP}}(\hat{\rho}) = \sum_n K_n\hat{\rho}K_n^\dagger$ and $\sum_n K_n^\dagger K_n = \mathbb{1}$, we have that

$$D(\hat{\rho}||\hat{\delta}) \geqslant D(\Phi_{\text{CPTP}}(\hat{\rho})||\Phi_{\text{CPTP}}(\hat{\delta}))$$

This is a generalisation of (C2a) and is true in the context of entanglement measures.

Proof. From the work of Vedral and Plenio [35], the case for the following argument is laid out in detail. They illustrate the fact that a complete measurement [36] may be written as a unitary operation, involving the partial trace, over an extended Hilbert space, $\mathcal{H} \otimes \mathcal{H}_n$, where the dimension of $\mathcal{H}_n$ is n. Let $\{|e_i\rangle\}$ be an orthonormal basis spanning $\mathcal{H}_n$, $|u\rangle$ be a unit vector in $\mathcal{H}_n$ and $\{K_i\}$ be a set of Kraus operators acting in $\mathcal{H}$. Using these, we may define the following operator [37], W.

$$W = \sum_i K_i \otimes |e_i\rangle\langle u|$$

Then $W^\dagger W = \mathbb{1} \otimes P_u$, where the projection P_u is given by $P_u = |u\rangle\langle u|$. This tells us that there exists a unitary operator U over $\mathcal{H} \otimes \mathcal{H}_n$, such that $W = U(\mathbb{1} \otimes P_u)$. This is because a unitary operator U satisfies $U^\dagger U = \mathbb{1}$. As a result, we can allow these aforementioned operators to act on a Hermitian matrix A on $\mathcal{H}$ in the following way.

$$U(A \otimes P_u)U^\dagger = \sum_{i,j} K_i A K_j^\dagger \otimes |e_i\rangle\langle e_j| \tag{1}$$

We may now apply the partial trace, using the definition given in [38], to the above operation. The result should now look familiar.

$$\mathrm{tr}_2 \left\{ U(A \otimes P_u)U^\dagger \right\} = \sum_i K_i A K_i^\dagger \tag{2}$$

This is the Kraus decomposition of an operator acting on A. If a distance measure satisfies $D(\mathrm{tr}_p\hat{\rho}||\mathrm{tr}_p\hat{\rho}) \leqslant D(\hat{\rho}||\hat{\delta})$ and $D(U\hat{\rho}U^\dagger||U\hat{\delta}U^\dagger) = D(\hat{\rho}||\hat{\delta}) = D(\hat{\rho} \otimes P_u||\hat{\delta} \otimes P_u)$, then the following set of statements are true.

$$\begin{aligned} D\left(\sum_i K_i\hat{\rho}K_i^\dagger || \sum_i K_i\hat{\delta}K_i^\dagger\right) &= D\left(\mathrm{tr}_2\left\{U(\hat{\rho} \otimes P_u)U^\dagger\right\} || \,\mathrm{tr}_2\left\{U(\hat{\delta} \otimes P_u)U^\dagger\right\}\right) \\ &\leqslant D\left(U(\hat{\rho} \otimes P_u)U^\dagger \,||\, U(\hat{\delta} \otimes P_u)U^\dagger\right) \\ &= D\left(\hat{\rho} \otimes P_u \,||\, \hat{\delta} \otimes P_u\right) \;=\; D(\hat{\rho} \,||\, \hat{\delta}) \end{aligned}$$

■

The above result proves (C2a), provided the assumptions are found to be true. This inspired me to use the set of properties regarding entanglement measures found in [35] for our investigation of coherence measures. We can state a corollary that can be extrapolated from the above result.

Corollary 4.3. It can be shown that, for a complete set of orthonormal projectors $\{P_n\}$, $\hat{\rho} \mapsto \sum_n P_n\hat{\rho}P_n$ is a CPTP map. Hence, by Theorem 4.4, the following result holds.

$$D(\hat{\rho}||\hat{\delta}) \geqslant D(\sum_n P_n\hat{\rho}P_n|| \sum_n P_n\hat{\delta}P_n))$$

We want the distance measure in question to also satisfy the following theorem.

Theorem 4.4. If a distance measure $D(\hat{\rho}||\hat{\delta})$ satisfies all of the above assumptions and $D(\sum_n P_n\hat{\rho}P_n||\sum_n P_n\hat{\delta}P_n)) = \sum_n D(P_n\hat{\rho}P_n||P_n\hat{\delta}P_n))$, then

$$D(\hat{\rho}||\hat{\delta}) \geqslant \sum_n D(K_n\hat{\rho}K_n^\dagger||K_n\hat{\delta}K_n^\dagger))$$

Proof. From the previous proof, we use Equations 1 and 2 again. We can use the argument given for Equation 2 to state the subsequent equality. For $P_i = |i\rangle\langle i|$, we have that

$$\mathrm{tr}_2\left\{\mathbb{1}\otimes P_i\ U(A\otimes P_u)U^\dagger\ \mathbb{1}\otimes P_i\right\} = K_iAK_i^\dagger$$

Our distance measure $D(\hat{\rho}||\hat{\delta})$ satisfies Corollary 4.3, decreases under partial tracing and is invariant under a tensor product with an orthonormal projector. This means that we can proceed in the following way.

$$\sum_i D\left(\mathrm{tr}_2\left\{\mathbb{1}\otimes P_i\ U(\hat{\rho}\otimes P_u)U^\dagger\ \mathbb{1}\otimes P_i\right\}||\mathrm{tr}_2\left\{\mathbb{1}\otimes P_i\ U(\hat{\delta}\otimes P_u)U^\dagger\ \mathbb{1}\otimes P_i\right\}\right)$$
$$\leqslant \sum_i D\left(\mathbb{1}\otimes P_i\ U(\hat{\rho}\otimes P_u)U^\dagger\ \mathbb{1}\otimes P_i||\mathbb{1}\otimes P_i\ U(\hat{\delta}\otimes P_u)U^\dagger\ \mathbb{1}\otimes P_i\right)$$
$$\leqslant D\left(U(\hat{\rho}\otimes P_u)U^\dagger||U(\hat{\delta}\otimes P_u)U^\dagger\right) = D\left(\hat{\rho}\otimes P_u||\hat{\delta}\otimes P_u\right) = D(\hat{\rho}\ ||\ \hat{\delta})$$

From this result, and the assumption that $\sum_i p_i D\left(\frac{\hat{\rho}_i}{p_i}||\frac{\hat{\delta}_i}{q_i}\right) \leqslant \sum_i D(\hat{\rho}_i||\hat{\delta}_i)$, we conclude:

$$\sum_i p_i D\left(\frac{\hat{\rho}_i}{p_i}\middle|\middle|\frac{\hat{\delta}_i}{q_i}\right) \leqslant D(\hat{\rho}||\hat{\delta})$$

■

Thus, proving the satisfaction of (C2b), and hence (C2a), whittles down to proving these properties of the functional. We have assumed the satisfaction of various characteristics in both proofs. For completeness, I shall list them all below.

(F2): $D(U\hat{\rho}U^\dagger||U\hat{\delta}U^\dagger) = D(\hat{\rho}||\hat{\delta})$, for any unitary operator, U.

(F3): $D(\mathrm{tr}_p\ \hat{\rho}||\mathrm{tr}_p\ \hat{\delta}) \leqslant D(\hat{\rho}||\hat{\delta})$, where tr_p is the partial trace.

(F4): $\sum_i p_i D\left(\frac{\hat{\rho}_i}{p_i}\middle|\middle|\frac{\hat{\delta}_i}{q_i}\right) \leqslant \sum_i D(\hat{\rho}_i||\hat{\delta}_i)$, for probabilities p_i and q_i.

(F5a): $D(\sum_i P_i\hat{\rho}P_i || \sum_i P_i\hat{\delta}P_i)) = \sum_i D(P_i\hat{\rho}P_i||P_i\hat{\delta}P_i))$

(F5b): $D\left(\hat{\rho}\otimes P_u \,||\, \hat{\delta}\otimes P_u\right) \;=\; D(\hat{\rho}\,||\,\hat{\delta})$, for $P_u = |u\rangle\langle u|$

Note that these properties are taken from [35].

We have just seen that it is sufficient to prove these properties, in order to show the satisfaction of (C2b) for functionals in the form of distance measures. We can show that the relative entropy satisfies these properties too.

(F2): We may use an alternate definition of relative entropy, in terms of the eigenvalues of $\hat{\rho}$ and $\hat{\delta}$ [39]. This is given by

$$C_{\text{Rel Ent.}}(\hat{\rho}) = \min_{\hat{\delta}\in I} \text{ tr}\left(\hat{\rho}\log\hat{\rho} - \hat{\rho}\log\hat{\delta}\right)$$

$$= -\sum_i \lambda_i \log\lambda_i + \sum_j \eta_j \log\eta_j$$

where λ_i and η_i are the eigenvalues of $\hat{\rho}$ and $\hat{\delta}$ respectively. Thus, it suffices to prove that the eigenvalues of $\hat{\rho}$ and $\hat{\delta}$ are invariant under unitary operations. By the Spectral theorem, we may always find an orthonormal basis, $\{|x\rangle\}$ such that the following is true.

$$U\hat{\rho}U^\dagger = U\sum_x \lambda_x|x\rangle\langle x|U^\dagger = \sum_x \lambda_x|x_U\rangle\langle x_U|$$

For another orthonormal basis $\{|x_U\rangle\}$ such that $U|x\rangle = |x_U\rangle$. As a result, the eigenvalues of a density matrix remain invariant under a unitary transformation. Consequently, we can proclaim that the relative entropy is similarly invariant.

(F3): Consider a quantum state, $\hat{\rho} = \hat{\rho}_1\otimes\hat{\rho}_2$, on a Hilbert space $\mathcal{H} = \mathcal{H}_1\otimes\mathcal{H}_2$. Let $\hat{\rho}_1 = \text{tr}_{\mathcal{H}_2}\hat{\rho}$. Then, we may compute $C_{\text{Rel Ent.}}(\hat{\rho}) = C_{\text{Rel Ent.}}(\hat{\rho}_1\otimes\hat{\rho}_2)$.

$$C_{\text{Rel Ent.}}(\hat{\rho}_1\otimes\hat{\rho}_2) = \text{tr}[(\hat{\rho}_1\otimes\hat{\rho}_2)\log(\hat{\rho}_1\otimes\hat{\rho}_2) - (\hat{\rho}_1\otimes\hat{\rho}_2)\log(\hat{\delta}_1\otimes\hat{\delta}_2)]$$

We may manipulate an identity given by Brewer [42] to achieve the following identity: $\log(A\otimes B) = \log A\otimes\mathbb{1} + \mathbb{1}\otimes\log B$. We can now use this identity in our equation for $C(\hat{\rho}_1\otimes\hat{\rho}_2)$.

$$\begin{aligned} C_{\text{Rel Ent.}}(\hat{\rho}_1\otimes\hat{\rho}_2) = \text{tr}[&(\hat{\rho}_1\otimes\hat{\rho}_2)(\log\hat{\rho}_1\otimes\mathbb{1} + \mathbb{1}\otimes\log\hat{\rho}_2) \\ &- (\hat{\rho}_1\otimes\hat{\rho}_2)(\log\hat{\delta}_1\otimes\mathbb{1} + \mathbb{1}\otimes\log\hat{\delta}_2] \end{aligned}$$

We can then proceed by using the mixed-product property of the tensor product, $(A\otimes B)(C\otimes D) = (AB)\otimes(CD)$, along with an appropriate factorisation.

$$
\begin{aligned}
&= \mathrm{tr}[(\hat{\rho}_1 \log \hat{\rho}_1 - \hat{\rho}_1 \log \hat{\delta}_1) \otimes \hat{\rho}_2 + \hat{\rho}_1 \otimes (\hat{\rho}_2 \log \hat{\rho}_2 - \hat{\rho}_2 \log \hat{\delta}_2)] \\
&= \mathrm{tr}[(\hat{\rho}_1 \log \hat{\rho}_1 - \hat{\rho}_1 \log \hat{\delta}_1)] \cdot \mathrm{tr}[\hat{\rho}_2] + \mathrm{tr}[\hat{\rho}_1] \cdot \mathrm{tr}[(\hat{\rho}_2 \log \hat{\rho}_2 - \hat{\rho}_2 \log \hat{\delta}_2)] \\
&= C_{\text{Rel Ent.}}(\hat{\rho}_1) \cdot \mathrm{tr}[\hat{\rho}_2] + \mathrm{tr}[\hat{\rho}_1] \cdot C_{\text{Rel Ent.}}(\hat{\rho}_2) \;\; = \;\; C_{\text{Rel Ent.}}(\hat{\rho}_1) + C_{\text{Rel Ent.}}(\hat{\rho}_2)
\end{aligned}
$$

Since $\hat{\rho}_1$ and $\hat{\rho}_2$ are density matrices, they are both of trace one. Thus, the last equality is trivial. We can now calculate $C(\mathrm{tr}_{\mathcal{H}_2}\{\hat{\rho}_1 \otimes \hat{\rho}_2\})$.

$$
\begin{aligned}
C_{\text{Rel Ent.}}(\mathrm{tr}_{\mathcal{H}_2}\{\hat{\rho}_1 \otimes \hat{\rho}_2\}) = \mathrm{tr}[&(\mathrm{tr}_{\mathcal{H}_2}\{\hat{\rho}_1 \otimes \hat{\rho}_2\}) \log(\mathrm{tr}_{\mathcal{H}_2}\{\hat{\rho}_1 \otimes \hat{\rho}_2\}) \\
&- (\mathrm{tr}_{\mathcal{H}_2}\{\hat{\rho}_1 \otimes \hat{\rho}_2\}) \log(\mathrm{tr}_{\mathcal{H}_2}\left\{\hat{\delta}_1 \otimes \hat{\delta}_2\right\})]
\end{aligned}
$$

By our assumption, $\hat{\rho}_1 = \mathrm{tr}_{\mathcal{H}_2}\hat{\rho}$. Thus, we can substitute this in to the equation above.

$$
C_{\text{Rel Ent.}}(\mathrm{tr}_{\mathcal{H}_2}\{\hat{\rho}_1 \otimes \hat{\rho}_2\}) = \mathrm{tr}[(\hat{\rho}_1 \log \hat{\rho}_1 - \hat{\rho}_1 \log \hat{\delta}_1)] \;\; = \;\; C_{\text{Rel Ent.}}(\hat{\rho}_1)
$$

We have proven that the relative entropy is always non-negative. Therefore, $C(\hat{\rho}_1) + C(\hat{\rho}_2) \geqslant C(\hat{\rho}_1)$. As a result, $C(\hat{\rho}) \geqslant C(\mathrm{tr}_{\mathcal{H}_2}\{\hat{\rho}\})$. A similar proof can be shown for the consideration of the alternative choice of partial trace, $\mathrm{tr}_{\mathcal{H}_1}$. It is completely analogous to this argument. Hence, the relative entropy of coherence decreases under partial tracing.

(F4): Here, we may use the axioms of the logarithm [43] in the following way.

$$
\begin{aligned}
&\sum_i D(\hat{\rho}_i || \hat{\delta}_i) = \mathrm{tr}\left(\hat{\rho}_i \log \hat{\rho}_i - \hat{\rho}_i \log \hat{\delta}_i\right) \\
&= \sum_i p_i \cdot \mathrm{tr}\left(\frac{\hat{\rho}_i}{p_i} \log\left(\frac{\hat{\rho}_i}{p_i}\right) - \frac{\hat{\rho}_i}{p_i} \log\left(\frac{\hat{\delta}_i}{q_i}\right)\right) + \sum_i p_i \cdot \mathrm{tr}\left(\log p_i - \log q_i\right) \\
&= \sum_i p_i \cdot \mathrm{tr}\left(\frac{\hat{\rho}_i}{p_i} \log\left(\frac{\hat{\rho}_i}{p_i}\right) - \frac{\hat{\rho}_i}{p_i} \log\left(\frac{\hat{\delta}_i}{q_i}\right)\right) + \sum_i p_i \log\left(\frac{p_i}{q_i}\right)
\end{aligned}
$$

We know that $p_i \log\left(\frac{p_i}{q_i}\right) \geqslant 0$. Therefore, we can assert that

$$
\mathrm{tr}\left(\hat{\rho}_i \log \hat{\rho}_i - \hat{\rho}_i \log \hat{\delta}_i\right) \leqslant \sum_i p_i \cdot \mathrm{tr}\left(\frac{\hat{\rho}_i}{p_i} \log\left(\frac{\hat{\rho}_i}{p_i}\right) - \frac{\hat{\rho}_i}{p_i} \log\left(\frac{\hat{\delta}_i}{q_i}\right)\right)
$$

Hence, (F4) is satisfied by the relative entropy of coherence.

(F5a): Let $\{P_i\}$ be a set of orthogonal projectors: $P_i \cdot P_j = \delta_{ij} P_i$ [44]. We know that the operation $\hat{\rho} \mapsto \sum_i P_i \hat{\rho} P_i$ diagonalises $\hat{\rho}$. Therefore, $\log(\sum_i P_i \hat{\rho} P_i) = \sum_i P_i \log(\hat{\rho}) P_i$. Then, we can spell this property of

(F4) out for the relative entropy.

$$D(\sum P_i\hat{\rho}P_i||\sum P_i\hat{\delta}P_i) =$$

$$= \mathrm{tr}\left(\sum_i P_i\hat{\rho}P_i \log\left(\sum_j P_j\hat{\rho}P_j\right) - \sum_i P_i\hat{\rho}P_i \log\left(\sum_l P_l\hat{\delta}P_l\right)\right)$$

$$= \mathrm{tr}\left(\sum_i P_i\hat{\rho}P_i \sum_j P_j \log\left(\hat{\rho}\right) P_j - \sum_i P_i\hat{\rho}P_i \sum_l P_l \log(\hat{\delta})P_l\right)$$

$$= \mathrm{tr}\left(\sum_i P_i\hat{\rho}P_iP_i \log\left(\hat{\rho}\right) P_i - \sum_i P_i\hat{\rho}P_iP_i \log(\hat{\delta})P_i\right)$$

$$= \sum_i \mathrm{tr}\left(P_i\hat{\rho}P_iP_i \log\left(\hat{\rho}\right) P_i - P_i\hat{\rho}P_iP_i \log(\hat{\delta})P_i\right) \;=\; \sum D(P_i\hat{\rho}P_i||P_i\hat{\delta}P_i)$$

Hence, we have just seen that the relative entropy satisfies (F5a).

(F5b): Recall that the projector P_u, known as a rank-1 projector, is given in terms of a unit vector $|u\rangle$, $P_u = |u\rangle\langle u|$. Hence it is of trace one. Also, it is important to recall that the trace function is multiplicative under a tensor product [45]. We may proceed with the following argument.

$$\begin{aligned} D(\hat{\rho}\otimes P_u||\hat{\delta}\otimes P_u) &= \mathrm{tr}\left((\hat{\rho}\otimes P_u)\log\left(\frac{\hat{\rho}\otimes P_u}{\hat{\delta}\otimes P_u}\right)\right) \\ &= \mathrm{tr}\left((\hat{\rho}\otimes P_u)\log\left(\frac{\hat{\rho}}{\hat{\delta}}\right)\right) \\ &= \mathrm{tr}\left(\hat{\rho}\log\left(\frac{\hat{\rho}}{\hat{\delta}}\right)\right)\mathrm{tr}(P_u) \\ &= \mathrm{tr}\left(\hat{\rho}\log\left(\frac{\hat{\rho}}{\hat{\delta}}\right)\right) = D(\hat{\rho}||\hat{\delta}) \end{aligned}$$

Thus, the relative entropy is invariant under an outer product with a single rank-1 projection operator.

All of these facts tell us that the relative entropy satisfies (C2b): Hence, the relative entropy is indeed a proper coherence measure. We shall see more from this functional later, when we begin to analyse the coherence of infinite-dimensional systems. Another well-documented measure is the l_1 norm of coherence. We shall discuss its properties below.

4.2. **Coherence in Terms of the Off-diagonal Elements.** The l_1 norm of coherence can be derived easily. We first start off with the l_1 norm of the quantum state in question, $\hat{\rho}$, minus its closest incoherent state. This would be the incoherent state that minimises the distance, as measured by a norm, between $\hat{\rho}$ and the set of incoherent states, I. It can be shown that the closest incoherent state to $\hat{\rho}$ is its diagonal form: $\hat{\rho}_{\text{diag}}$. Using Example 2.10, we get the following result.

$$C_{l_1}(\hat{\rho}) = \|\hat{\rho} - \hat{\rho}_{\text{diag}}\|_1 = \sum_{i,j} |\hat{\rho}_{ij} - \hat{\rho}_{ii}|$$

Therefore, we can write the l_1 norm of coherence in terms of the off-diagonal elements of the quantum state.

Definition 4.5. Given a quantum state $\hat{\rho}$, the l_1 norm of Coherence is defined as:

$$C_{l_1}(\hat{\rho}) = \sum_{i \neq j} |\hat{\rho}_{ij}|$$

Let us now discuss the properties of the l_1 norm of coherence: To see if it satisfies (C1)-(C3) and therefore be able to check whether it may be called a coherence measure. The first property, (C1), is trivial to prove.

(C1): Let $\hat{\delta} \in I$, then $\hat{\delta}$ is a density matrix, whose off-diagonal elements are all zero. Thus, we achieve the following result.

$$C_{l_1}(\hat{\delta}) = \sum_{i \neq j} |\hat{\delta}_{ij}| = \sum_{i \neq j} |0| = 0$$

As we did with the relative entropy, we shall now prove (C3) for the l_1 norm of coherence.

(C3): We wish to prove that the l_1 norm of coherence is convex using only two arguments: $\hat{\rho}_0$, $\hat{\rho}_1$, with probabilities p_0 and p_1. We then proceed in the following way.

$$\begin{aligned} C_{l_1}(p_0\hat{\rho}_0 + p_1\hat{\rho}_1) &= \sum_{i \neq j} |p_0[\hat{\rho}_0]_{ij} + p_1[\hat{\rho}_1]_{ij}| \\ &\leqslant \sum_{i \neq j} |p_0[\hat{\rho}_0]_{ij}| + \sum_{i \neq j} |p_1[\hat{\rho}_1]_{ij}| \\ &= p_0 \sum_{i \neq j} |[\hat{\rho}_0]_{ij}| + p_1 \sum_{i \neq j} |[\hat{\rho}_1]_{ij}| \\ &= p_0 \cdot C_{l_1}(\hat{\rho}_0) + p_1 \cdot C_{l_1}(\hat{\rho}_1) \end{aligned}$$

If we proceed by induction, we will see that the l_1 norm of coherence satisfies (C3). We must now prove that the l_1 norm of coherence satisfies (F2)-(F5), as per our argument for the relative entropy.

(F2): We can show that the l_1 norm of coherence is unitarily invariant.

$$C_{l_1}(U\hat{\rho}U^\dagger) = C_{l_1}(\hat{\rho})$$

To do this, we refer to the work of Watrous [46]. He states that l_p norms, for $p \in [1, \infty]$, are invariant under a unitary operation. Therefore, we must write the l_1 norm in its full form and apply a unitary operation as follows.

$$C_{l_1}(U\hat{\rho}U^\dagger) = \left\| U\hat{\rho}U^\dagger - U\hat{\rho}_{\text{diag}}U^\dagger \right\|_1$$

$$= \left\| U\left(\hat{\rho} - \hat{\rho}_{\text{diag}}\right)U^\dagger \right\|_1$$

$$= \|\hat{\rho} - \hat{\rho}_{\text{diag}}\|_1 \ = \ C_{l_1}(\hat{\rho})$$

Therefore, the l_1 norm of coherence is unitarily invariant.

(F3): One can show that the following bound holds for any l_p norm and any bounded linear operator A defined on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, by the work of Rastegin [47].

$$\|\text{tr}_{\mathcal{H}_1}\{A\}\|_p \leqslant [\dim(\mathcal{H}_1)]^{(p-1)/p}\|A\|_p$$

By setting $p = 1$ and $A = \hat{\rho} - \hat{\delta}$, we get the following inequality.

$$\|\text{tr}_{\mathcal{H}_1}\left\{\hat{\rho} - \hat{\delta}\right\}\|_1 \leqslant [\dim(\mathcal{H}_1)]\|\hat{\rho} - \hat{\delta}\|_1$$

Since $\dim(H_1) \geqslant 1$ is true for any chosen $\mathcal{H}_1$, this inequality always holds. Hence, the l_1 norm of coherence decreases under partial tracing. It is important to note that a similar argument can be made for the alternative choice of partial trace, $\text{tr}_{\mathcal{H}_2}$.

(F4): This is very straightforward to prove: we first calculate $\sum_i p_i C_{l_1}(\hat{\rho}_i/p_i)$.

$$\sum_i p_i \cdot C_{l_1}\left(\frac{\hat{\rho}_i}{p_i}\right) = \sum_i p_i \cdot \left\| \frac{\hat{\rho}_i}{p_i} - \frac{[\hat{\rho}_{\text{diag}}]_i}{p_i} \right\|_1$$

$$= \sum_i \|\hat{\rho}_i - [\hat{\rho}_{\text{diag}}]_i\|_1$$

$$= \sum_i C_{l_1}(\hat{\rho}_i)$$

So, in fact, we find that the l_1 norm of coherence satisfies a strict equality of (F4).

(F5a): We may use the longer form of $C_{l_1}(\hat{\rho})$ in order to prove this property.

$$C_{l_1}(\sum_i P_i \ \hat{\rho} \ P_i) = \left\| \sum_i P_i \ \hat{\rho} \ P_i - \sum_i P_i \ \hat{\rho}_{\text{diag}} \ P_i \right\|_1$$

As $\hat{\rho}_{\text{diag}}$ is already diagonalised, the projection operators have no effect on $\hat{\rho}_{\text{diag}}$. Thus, we may rewrite the above equation in the following way.

$$
\begin{aligned}
C_{l_1}(\sum_i P_i\ \hat{\rho}\ P_i) &= \left\| \sum_i (P_i\ \hat{\rho}\ P_i - P_i\ \hat{\rho}_{\text{diag}}\ P_i) \right\|_1 \\
&= \operatorname{tr}\left(\sqrt{\sum_i (P_i\ \hat{\rho}\ P_i - P_i\ \hat{\rho}_{\text{diag}}\ P_i)^\dagger \sum_i (P_i\ \hat{\rho}\ P_i - P_i\ \hat{\rho}_{\text{diag}}\ P_i)} \right) \\
&= \operatorname{tr}\left(\sqrt{\sum_i (P_i\ \hat{\rho}\ P_i - P_i\ \hat{\rho}_{\text{diag}}\ P_i)\,(P_i\ \hat{\rho}\ P_i - P_i\ \hat{\rho}_{\text{diag}}\ P_i)} \right) \\
&= \operatorname{tr}\left(\sqrt{\sum_i \left(P_i\ \hat{\rho}^2\ P_i - 2P_i\ \hat{\rho}\cdot\hat{\rho}_{\text{diag}}\ P_i + P_i\ \hat{\rho}^2_{\text{diag}}\ P_i\right)} \right) \\
&= \operatorname{tr}\left(\sqrt{\sum_i P_i \left(\ \hat{\rho}^2 - 2\hat{\rho}\cdot\hat{\rho}_{\text{diag}} + \hat{\rho}^2_{\text{diag}}\right) P_i} \right) \\
&= \operatorname{tr}\left(\sum_i P_i \sqrt{\left(\ \hat{\rho}^2 - 2\hat{\rho}\cdot\hat{\rho}_{\text{diag}} + \hat{\rho}^2_{\text{diag}}\right)} P_i \right) \\
&= \sum_i \operatorname{tr}\left(P_i \sqrt{\left(\ \hat{\rho}^2 - 2\hat{\rho}\cdot\hat{\rho}_{\text{diag}} + \hat{\rho}^2_{\text{diag}}\right)} P_i \right) \\
&= \sum_i \operatorname{tr}\left(P_i \sqrt{\left(\ \hat{\rho} - \hat{\rho}_{\text{diag}}\right)^2} P_i \right) \\
&= \sum_i \left\|(P_i\ \hat{\rho}\ P_i - P_i\ \hat{\rho}_{\text{diag}}\ P_i)\right\|_1 = \sum_i C_{l_1}(P_i\ \hat{\rho}\ P_i)
\end{aligned}
$$

Thus, we can take the summation outside when projective measurements operate on the l_1 norm of coherence.

(F5b): We can compute $C_{l_1}(\hat{\rho}\otimes P_u)$ explicitly.

$$
\begin{aligned}
C_{l_1}(\hat{\rho}\otimes P_u) &= \|\hat{\rho}\otimes P_u - \hat{\rho}_{\text{diag}}\otimes P_u\|_1 \\
&= \|\hat{\rho} - \hat{\rho}_{\text{diag}}\|_1 \cdot \|P_u\|_1 \\
&= \|\hat{\rho} - \hat{\rho}_{\text{diag}}\|_1 \\
&= C_{l_1}(\hat{\rho})
\end{aligned}
$$

Therefore, the l_1 norm of coherence is invariant under a tensor product with a rank-1 projection operator, P_u.

We have just seen that the l_1 norm of coherence is indeed a coherence measure, using the analogy between entanglement and coherence measures.

4.3. **Coherence in Terms of the Commutator.** As we know, we can quantify coherence in terms of how "off-diagonal" a quantum state is. Rather, we can study the coherence in terms of how much a density matrix, $\hat{\rho}$, differs from a diagonal matrix. One way in which we may strictly quantify this is by using the number operator, $\hat{N}$, defined as $\hat{N} = \sum_n |n\rangle\langle n|$. One can pose the question: How well does the chosen state, $\hat{\rho}$, commute with the number operator? As a result, we can observe how "diagonal" a chosen density matrix is. It was this line of thought which led to the construction of the following functional.

Definition 4.6. Given a quantum state $\hat{\rho}$ and the number operator $\hat{N} = \sum_n |n\rangle\langle n|$, the Commutator of Coherence, $C_{[\cdot,\cdot]}(\hat{\rho})$, as:

$$C_{[\cdot,\cdot]}(\hat{\rho}) = \left\| [\hat{\rho}, \hat{N}] \right\|_1$$

This should tell us how incoherent a state is, by evaluating how much it resembles a standard diagonal matrix. We shall now study whether $C_{[\cdot,\cdot]}(\hat{\rho})$ satisfies properties (C1)-(C3). It is trivial to see that (C1) is satisfied by this potential measure.

(C1): Let $\hat{\delta} \in I$, then $\hat{\delta}$ is a density matrix that commutes with any diagonal matrix. Therefore, $\hat{\delta}$ commutes with the number operator, i.e. $[\hat{\delta}, \hat{N}] = 0$. Thus, we achieve the following result.

$$C_{[\cdot,\cdot]}(\hat{\delta}) = \left\| [\hat{\delta}, \hat{N}] \right\|_1 = \|0\|_1 = 0$$

Hence, (C1) is satisfied by the Commutator of Coherence. We may see that (C3) is also satisfied.

(C3): The same set-up is carried out as before. Because of the linearity of the commutator, and the fact that the trace norm is indeed convex, we may proceed with the following argument.

$$\begin{aligned} C_{[\cdot,\cdot]}\Big(\sum_n p_n \hat{\rho}_n\Big) &= \left\| \Big[\sum_n p_n \hat{\rho}_n, \hat{N}\Big] \right\|_1 \\ &= \left\| \sum_n [p_n \hat{\rho}_n, \hat{N}] \right\|_1 \\ &= \left\| \sum_n p_n [\hat{\rho}_n, \hat{N}] \right\|_1 \\ &\leqslant \sum_n p_n \left\| [\hat{\rho}_n, \hat{N}] \right\|_1 = \sum_n p_n C_{[\cdot,\cdot]}(\hat{\rho}_n) \end{aligned}$$

Hence, $C_{[\cdot,\cdot]}(\hat{\rho})$ appeases (C3) and is therefore convex.

Unfortunately, the Commutator of Coherence violates (C2b). Rather than going through all of the sub-properties of (C2b), we shall just discuss the condition it breaches.

(F3): Let $\hat{\rho}_{AB}$ be any density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$. This can be examined, using the partial trace, as

$$\mathrm{tr}_B(\hat{\rho}_{AB}) = \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \langle b_l|b_k\rangle\,, \quad c_{ijkl} \in \mathbb{C}$$

We shall denote this as $\hat{\rho}_A := \mathrm{tr}_B(\hat{\rho}_{AB})$ on $\mathcal{H}_A$ of our described system. It is also important to note that $\langle b_l|b_k\rangle = \mathrm{tr}(|b_l\rangle\langle b_k|) \in \mathbb{C}$. We can apply this notation in order to investigate (F3) for $C_{[\cdot,\cdot]}(\hat{\rho})$.

$$\begin{aligned} C_{[\cdot,\cdot]}(\hat{\rho}_A) &= \left\| [\sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \langle b_l|b_k\rangle\,, \hat{N}] \right\|_1 \\ &= \left\| \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \langle b_l|b_k\rangle\, \hat{N} - \hat{N} \sum_{pqrs} c_{pqrs} |a_p\rangle\langle a_q| \langle b_r|b_s\rangle \right\|_1 \end{aligned}$$

This looks promising, as $0 \leqslant \langle b_l|b_k\rangle \leqslant 1$ (from the fact that density matrices are of trace one). Therefore, we might be able to prove $C_{[\cdot,\cdot]}(\hat{\rho}_{AB}) \geqslant C_{[\cdot,\cdot]}(\hat{\rho}_A)$ and, hence, show that $C_{[\cdot,\cdot]}(\hat{\rho})$ decreases under partial tracing. However, when we explicitly calculate $C_{[\cdot,\cdot]}(\hat{\rho}_{AB})$, we soon experience problems.

$$\begin{aligned} C_{[\cdot,\cdot]}(\hat{\rho}_{AB}) &= \left\| [\sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_l|, \hat{N}] \right\|_1 \\ &= \left\| \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_l| \hat{N} - \hat{N} \sum_{pqrs} c_{pqrs} |a_p\rangle\langle a_q| \otimes |b_r\rangle\langle b_s| \right\|_1 \\ &= \left\| \sum_{n} \sum_{ijkl} c_{ijkl} \langle b_l|n\rangle\, |a_i\rangle\langle a_j| \otimes |b_k\rangle\langle n| - \sum_{n} \sum_{pqrs} c_{pqrs} \langle n|a_p\rangle\, |n\rangle\langle a_q| \otimes |b_r\rangle\langle b_s| \right\|_1 \end{aligned}$$

In general, we are unable to factorise and simplify the last equality. We cannot expect anything well-behaved here. This is because the commutator is a Lie algebra [48], whereas the tensor product inside of a commutator is not a Lie algebra [49]. Hence, no smooth mapping exists to encode the transformation. More informally, we can also see this by considering the fact that, in general, $|a_i\rangle \neq |n\rangle$ and $\langle n| \neq \langle b_s|$ for all summation indices n, s and i.

It can be shown that the properties (F2), (F4), (F5a) and (F5b) are satisfied by $C_{[\cdot,\cdot]}(\hat{\rho})$. This is evidence that the method for constructing original coherence measures is difficult: A promising functional like this fails due to the fact that we cannot, in general, show that it decreases under partial tracing.

4.4. **The Fidelity of Quantum Coherence.** Another promising functional that fails upon inspection of (C2b) is the fidelity of quantum coherence defined below.

Definition 4.7. If there exists a maximally distant incoherent state $\hat{\delta}^* \in I$ to a given state $\hat{\rho}$, the Fidelity of Quantum Coherence, $C_F(\hat{\rho})$, is defined as:

$$C_F(\hat{\rho}) = 1 - \max_{\hat{\delta} \in I} F(\hat{\rho}, \hat{\delta}) = 1 - \mathrm{tr}(|\sqrt{\hat{\rho}}\sqrt{\hat{\delta}^*}|)$$

It can also be written in a different form:

$$C_F(\hat{\rho}) = 1 - \mathrm{tr}\left(\sqrt{\sqrt{\hat{\rho}}\ \hat{\delta}^*\ \sqrt{\hat{\rho}}}\right)$$

It can be shown that the coherence measure induced by fidelity satisfies (C1), (C2a) and (C3). However, it violates (C2b). We shall provide an original counter-example that disproves (C2b) for the measure induced by fidelity. We proceed by contradiction.

Recall, we require that the fidelity of quantum coherence satisfies the following inequality: $C_F(\hat{\rho}) \geqslant \sum_n p_n C_F(\hat{\rho}_n)$. This may be written as $C_F(\hat{\rho}) \geqslant \sum_n p_n C_F(K_n \hat{\rho} K_n^\dagger / p_n)$. I give the non-trivial two-dimensional Kraus operators below.

$$K_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} |a|^2 = 1 \\ |b|^2 + |c|^2 = 1 \end{matrix}$$

Using the conditions on the right, it is clear to see that the set of Kraus operators $\{K_1, K_2\}$ satisfy all of the necessary requirements upheld by Kraus operators. The two-dimensional quantum state, $\hat{\rho}$, and an incoherent state yet to be determined, $\hat{\delta}$, are given below.

$$\hat{\rho} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \hat{\delta} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

We can now calculate $\hat{\rho}_1 = K_1 \hat{\rho} K_1^\dagger$ and $\hat{\rho}_2 = K_2 \hat{\rho} K_2^\dagger$. We shall first consider the term $p_1 C_F(\hat{\rho}_1)$.

$$\hat{\rho}_1 = K_1 \hat{\rho} K_1^\dagger = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{b} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{a\bar{b}}{4} \\ \frac{\bar{a}b}{4} & \frac{|b|^2}{2} \end{pmatrix}$$

We may take the square root of this 2×2 matrix using standard methods given by Levinger [50].

$$\sqrt{\hat{\rho}_1} = \sqrt{K_1 \hat{\rho} K_1^\dagger} = \frac{\sqrt{2}}{\sqrt{1 + |b|^2 + \sqrt{3}|b|}} \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}|b|}{4} & \frac{a\bar{b}}{4} \\ \frac{\bar{a}b}{4} & \frac{|b|^2}{2} + \frac{\sqrt{3}|b|}{4} \end{pmatrix}$$

Now that we have the square root of $\hat{\rho}_1$, we may compute $p_1 C_F(\hat{\rho}_1)$.

$$p_1 C_F(\hat{\rho}_1) = p_1 - \mathrm{tr}\left(\sqrt{\sqrt{K_1 \hat{\rho} K_1^\dagger}\ \hat{\delta}\ \sqrt{K_1 \hat{\rho} K_1^\dagger}}\right)$$

$$= p_1 - \frac{2}{1 + |b|^2 + \sqrt{3}|b|} \times$$

$$\mathrm{tr}\left(\sqrt{\begin{pmatrix} x\left(\frac{1}{2}+\frac{\sqrt{3}|b|}{4}\right)^2 + \frac{|b|^2}{16}y & \frac{a\bar{b}}{4}\left[x\left(\frac{1}{2}+\frac{\sqrt{3}|b|}{4}\right) + y\left(\frac{|b|^2}{2}+\frac{\sqrt{3}|b|}{4}\right)\right] \\ \frac{\bar{a}b}{4}\left[x\left(\frac{1}{2}+\frac{\sqrt{3}|b|}{4}\right) + y\left(\frac{|b|^2}{2}+\frac{\sqrt{3}|b|}{4}\right)\right] & y\left(\frac{|b|^2}{2}+\frac{\sqrt{3}|b|}{4}\right)^2 + \frac{|b|^2}{16}x \end{pmatrix}}\right)$$

This matrix may look confusing at first. However, when we apply the standard methods discussed previously, we may find the trace of this rather straightforwardly. This result can be found below.

$$p_1C_F(\hat{\rho}_1) = p_1 - \frac{2}{1+|b|^2+\sqrt{3}|b|}\times$$

$$\sqrt{2\sqrt{xy}\left[\left(\frac{1}{2}+\frac{\sqrt{3}|b|}{4}\right)\left(\frac{|b|^2}{2}+\frac{\sqrt{3}|b|}{4}\right)-\frac{|b|^2}{16}\right]+x\left(\frac{1}{2}+\frac{\sqrt{3}|b|}{4}\right)^2+y\left(\frac{|b|^2}{2}+\frac{\sqrt{3}|b|}{4}\right)^2+\frac{|b|^2}{16}}$$

We shall now proceed to find $p_2C_F(\hat{\rho}_2)$: Fortunately, this is a lot more trivial.

$$\sqrt{\hat{\rho}_2} = \sqrt{K_2\hat{\rho}K_2^\dagger} = \sqrt{\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \bar{c} & 0 \end{pmatrix}} = \begin{pmatrix} \frac{|c|}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}$$

From this, we can work out $p_2C_F(\hat{\rho}_2)$ rather easily.

$$\begin{aligned} p_2C_F(\hat{\rho}_2) &= p_2 - \mathrm{tr}\left(\sqrt{\sqrt{K_2\hat{\rho}K_2^\dagger}\ \hat{\delta}\ \sqrt{K_2\hat{\rho}K_2^\dagger}}\right) \\ &= p_2 - \mathrm{tr}\left(\sqrt{\begin{pmatrix} \frac{|c|}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} \frac{|c|}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}}\right) \\ &= p_2 - \frac{|c|}{\sqrt{2}}\sqrt{x} \end{aligned}$$

It can be shown that $C_F(\hat{\rho})$, in our example, is given by the following equation. For the much needed sake of brevity, I have not included the calculations behind the result.

$$C_F(\hat{\rho}) = 1 - \frac{1}{1+\frac{\sqrt{3}}{2}}\sqrt{2\sqrt{xy}\left[\left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right)-\frac{1}{16}\right]+(x+y)\left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right)^2+\frac{1}{16}}$$

In order to make this example easier to follow, I shall choose a maximally distant incoherent state, as I compute the fidelity. That is, I shall choose $x=0$ and $y=1$. This choice is still allowed by the definition of an incoherent state in the set I. We shall also let $|b|$, one of the elements of K_1, be very small so that $|b|^2 \approx 0$. Thus, we may proceed with the following argument, using our new conditions for x, y and $|b|$.

$$C_F(\hat{\rho}) = 1 - \frac{1}{1+\frac{\sqrt{3}}{2}}\sqrt{\left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right)^2+\frac{1}{16}}$$

$$\approx 1 - 0.517776$$

From our previous calculations, we may compute $\sum_n p_nC_F(\hat{\rho}_n)$ for $n \in [1,2]$. Simultaneously substituting in our restrictions on x, y and $|b|$, we achieve the following, simple result.

$$\sum_n p_n C_F(\hat{\rho}_n) = p_1 C_F(\hat{\rho}_1) + p_2 C_F(\hat{\rho}_2)$$
$$= p_1 + p_2 = 1$$

Therefore, we can claim an important result (specific to this example, of course!).

$$\sum_n p_n C_F(\hat{\rho}_n) = 1 \quad > \quad 1 - 0.517776 \approx C_F(\hat{\rho})$$

This tells us all we need to know: The fidelity of coherence is non-decreasing under an example of subselective measurements. Hence, (C2b) is not satisfied, in general, for the distance measure induced by the Fidelity. This is an original proof: Another (much more concise) argument that uses Bloch representation of quantum states was laid out by Shao et. al. [51]. We may interpret this argument to say that the fidelity of quantum coherence is not a suitable coherence measure.

After seeing two functionals that failed to be coherence measures, it would be timely to discuss a very promising potential measure of coherence.

4.5. **Distance Traversed by the Dephasing Map.** In this discussion, we shall refer back to Definition 2.4 for the definition of the dephasing map. We can construct a potential coherence measure using this map.

Definition 4.8. Given a quantum state $\hat{\rho}$, the Dephasing Distance of Coherence is defined as:

$$C_\Phi(\hat{\rho}) = \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} - \Phi(\hat{\rho})|\right)$$

where $\Phi(\hat{\rho})$, the dephasing map, is given by $\Phi(\hat{\rho}) = \sum_n |n\rangle\langle n|\hat{\rho}|n\rangle\langle n|$

The intuition behind this functional lies in the fact that $\Phi(\hat{\rho})$ maps $\hat{\rho}$ to its closest incoherent form. This can be seen by considering the Hilbert-Schmidt norm [52] of $\hat{\rho} - \Phi(\hat{\rho})$. Then, we can show that this distance is minimised.

The Hilbert-Schmidt norm of a matrix A can be written in terms of its singular values, $\{s_n\}$: $\|A\|_{\mathrm{HS}} = (\sum_{n\geqslant 1} s_n^2(A))^{\frac{1}{2}}$. The singular values of A can be calculated by square rooting the eigenvalues of the self-adjoint non-negative operator $A^\dagger A$. In order to see that this is minimised, we shall examine the square of the Hilbert-Schmidt distance between $\hat{\rho}$ and $\Phi(\hat{\rho})$. We first use the fact that the square can be expanded.

$$\|\hat{\rho} - \Phi(\hat{\rho})\|_{\mathrm{HS}}^2 = \|\hat{\rho}\|_{\mathrm{HS}}^2 - 2\,\|\hat{\rho}\cdot\Phi(\hat{\rho})\|_{\mathrm{HS}} + \|\Phi(\hat{\rho})\|_{\mathrm{HS}}^2$$
$$\geqslant \|\hat{\rho}\|_{\mathrm{HS}}^2 - 2\,\|\Phi(\hat{\rho})\|_{\mathrm{HS}}\cdot\|\hat{\rho}\|_{\mathrm{HS}} + \|\Phi(\hat{\rho})\|_{\mathrm{HS}}^2$$

We achieve a lower bound from the fact that the Hilbert-Schmidt norm is submultiplicative [53]. That means that $\|\hat{\rho}\cdot\Phi(\hat{\rho})\|_{\mathrm{HS}} \leqslant \|\Phi(\hat{\rho})\|_{\mathrm{HS}}\cdot\|\hat{\rho}\|_{\mathrm{HS}}$.

Hence, in order to show that the Hilbert-Schmidt norm is minimised we must prove that the inequality is saturated: $\|\hat{\rho}\cdot\Phi(\hat{\rho})\|_{\text{HS}} = \|\Phi(\hat{\rho})\|_{\text{HS}}\cdot\|\hat{\rho}\|_{\text{HS}}$.

$$\begin{aligned}\|\Phi(\hat{\rho})\|_{\text{HS}}\cdot\|\hat{\rho}\|_{\text{HS}} &= \left(\sum_{i\geqslant 1} s_i^2(\hat{\rho})\right)^{\frac{1}{2}} \times \left(\sum_{j\geqslant 1}\sum_{n} |n\rangle\langle n|s_j^2(\hat{\rho})|n\rangle\langle n|\right)^{\frac{1}{2}} \\ &= \left(\sum_{i\geqslant 1} s_i^2(\hat{\rho})\right)^{\frac{1}{2}} \times \left(\sum_{j\geqslant 1} s_j^2(\hat{\rho})\right)^{\frac{1}{2}} = \left(\sum_{i\geqslant 1} s_i^2(\hat{\rho})\sum_{j\geqslant 1} s_j^2(\hat{\rho})\right)^{\frac{1}{2}}\end{aligned}$$

$$\|\Phi(\hat{\rho})\cdot\hat{\rho}\|_{\text{HS}} = \left(\sum_{i\geqslant 1} s_i^2(\hat{\rho})\sum_{j\geqslant 1}\sum_{n} |n\rangle\langle n|s_j^2(\hat{\rho})|n\rangle\langle n|\right)^{\frac{1}{2}} = \left(\sum_{i\geqslant 1} s_i^2(\hat{\rho})\sum_{j\geqslant 1} s_j^2(\hat{\rho})\right)^{\frac{1}{2}}$$

Hence, $\|\hat{\rho}\cdot\Phi(\hat{\rho})\|_{\text{HS}} = \|\Phi(\hat{\rho})\|_{\text{HS}}\cdot\|\hat{\rho}\|_{\text{HS}}$ as the dephasing map leaves the singular values of $\hat{\rho}$ invariant. Thus, we can say that the dephasing map minimises the Hilbert-Schmidt distance. Therefore, the distance measure we constructed could potentially be a good coherence measure, as it measures from the closest incoherent state, $\Phi(\hat{\rho})$, to $\hat{\rho}$.

We may now investigate (C1)-(C3) for this potential measure. They follow in a similar vein to the l_1 norm of coherence.

(C1): Let $\hat{\delta}\in I$, then $\hat{\delta}$ is a density matrix, whose off-diagonal elements are all zero. We use the fact that the dephasing map leaves a diagonal matrix invariant. Thus, we achieve the following result.

$$C_\Phi(\hat{\delta}) = \frac{1}{2}\text{tr}\left(|\hat{\delta}-\Phi(\hat{\delta})|\right) = \frac{1}{2}\text{tr}\left(|\hat{\delta}-\hat{\delta}|\right) = 0$$

We can now turn to (C3) for this functional. It is fairly easy to see that (C3) is satisfied from the fact that the trace norm is convex.

(C3): By the definition of $C_\Phi(\hat{\rho})$ and the usual set-up for the proof of (C3), we see that the following argument holds.

$$\begin{aligned} C_\Phi(\sum_n p_n\hat{\rho}_n) &= \frac{1}{2}\text{tr}\left(|\sum_n p_n\hat{\rho}_n - \Phi(\sum_n p_n\hat{\rho}_n)|\right) \\ [\Phi(\hat{\rho}) \text{ is linear} \Rightarrow] \quad &= \frac{1}{2}\text{tr}\left(|\sum_n p_n\hat{\rho}_n - \sum_n p_n\Phi(\hat{\rho}_n)|\right) \\ &= \frac{1}{2}\text{tr}\left(|\sum_n p_n\left(\hat{\rho}_n - \Phi(\hat{\rho}_n)\right)|\right) \\ [\text{tr}\left(|\cdot|\right) \text{ is convex} \Rightarrow] \quad &\leqslant \frac{1}{2}\sum_n p_n\text{tr}\left(|\left(\hat{\rho}_n - \Phi(\hat{\rho}_n)\right)|\right) \\ &= \sum_n p_n C_\Phi(\hat{\rho}_n)\end{aligned}$$

Therefore, $C_\Phi(\hat{\rho})$ is non-increasing under mixing of states. In other words, $C_\Phi(\hat{\rho})$ is convex.

We shall now examine the sub-properties of (C2b) as before. If the functional satisfies (F2)-(F5), then it satisfies (C2b) by the argument mentioned in the discussion of relative entropy.

(F2): This is fairly starightforward to prove. By the argument given by Watrous [46], we can proceed in the following way.

$$C_\Phi(U\hat{\rho}U^\dagger) = \frac{1}{2}\text{tr}\left(|U\hat{\rho}U^\dagger - \Phi(U\hat{\rho}U^\dagger)|\right)$$

$$= \frac{1}{2}\text{tr}\left(|U\hat{\rho}U^\dagger - U\Phi(\hat{\rho})U^\dagger|\right)$$

$$= \frac{1}{2}\text{tr}\left(|U(\hat{\rho} - \Phi(\hat{\rho}))U^\dagger|\right)$$

$$= \frac{1}{2}\text{tr}\left(\sqrt{(U(\hat{\rho} - \Phi(\hat{\rho}))U^\dagger)^\dagger\,(U(\hat{\rho} - \Phi(\hat{\rho}))U^\dagger)}\right)$$

$$= \frac{1}{2}\text{tr}\left(\sqrt{U(\hat{\rho} - \Phi(\hat{\rho}))^\dagger(\hat{\rho} - \Phi(\hat{\rho}))U^\dagger}\right) = \frac{1}{2}\text{tr}\left(|U^\dagger(\hat{\rho} - \Phi(\hat{\rho}))|\right)$$

$$= \frac{1}{2}\text{tr}\left(\sqrt{(U^\dagger(\hat{\rho} - \Phi(\hat{\rho})))^\dagger\,(U^\dagger(\hat{\rho} - \Phi(\hat{\rho})))}\right)$$

$$= \frac{1}{2}\text{tr}\left(\sqrt{(\hat{\rho} - \Phi(\hat{\rho}))^\dagger UU^\dagger(\hat{\rho} - \Phi(\hat{\rho}))}\right)$$

$$= \frac{1}{2}\text{tr}\left(\sqrt{(\hat{\rho} - \Phi(\hat{\rho}))^\dagger(\hat{\rho} - \Phi(\hat{\rho}))}\right)$$

$$= \frac{1}{2}\text{tr}\left(|\hat{\rho} - \Phi(\hat{\rho})|\right) = C_\Phi(\hat{\rho})$$

Therefore, $C_\Phi(\hat{\rho})$ is invariant under a unitary transformation, described by a unitary matrix U and its inverse $U^\dagger$.

(F3): Consider a Hilbert Space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $\hat{\rho}_1 = \text{tr}_{\mathcal{H}_2}\hat{\rho}$. Then, we may compute $C_\Phi(\hat{\rho}_1)$.

$$C_\Phi(\hat{\rho}_1) = \frac{1}{2}\text{tr}\left(|\hat{\rho}_1 - \Phi(\hat{\rho}_1)|\right) = \frac{1}{2}\text{tr}\left(|\text{tr}_{\mathcal{H}_2}\hat{\rho} - \Phi(\text{tr}_{\mathcal{H}_2}\hat{\rho})|\right)$$

As $\hat{\rho}$ is defined over $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, we can suggest that $\hat{\rho} \geqslant \hat{\rho}_1$.This is because either all of the information encoded in $\hat{\rho}$ belongs to $\hat{\rho}_1$ or the information is shared between $\hat{\rho}_1$ and $\hat{\rho}_2$. Therefore, by the subadditivity property, we have that

$$C_\Phi(\hat{\rho}_1) = \frac{1}{2}\mathrm{tr}\left(|\hat{\rho}_1 - \Phi(\hat{\rho}_1)|\right) \quad \leqslant \quad \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} - \Phi(\hat{\rho})|\right) = C_\Phi(\hat{\rho})$$

Hence, the dephasing distance of coherence decreases under partial tracing.

(F4): This property involves another straightforward proof. We may write our functional in terms of a distance measure: $D(\hat{\rho}||\Phi(\hat{\rho}))$. We actually achieve a strict equality for this property.

$$\begin{aligned}\sum_i p_i D\left(\frac{\hat{\rho}_i}{p_i}\middle|\middle|\frac{\Phi(\hat{\rho}_i)}{p_i}\right) &= \sum_i p_i \mathrm{tr}\left(\left|\frac{\hat{\rho}_i}{p_i} - \frac{\Phi(\hat{\rho}_i)}{p_i}\right|\right)\\ &= \sum_i \mathrm{tr}\left(\left|\hat{\rho}_i - \Phi(\hat{\rho}_i)\right|\right)\\ &= \sum_i D(\hat{\rho}_i||\Phi(\hat{\rho}_i))\end{aligned}$$

(F5a): In order to prove this property, we have to first write our functional in a different way. It can be shown that $C(\Phi(\hat{\rho}))$ can be written as the supremum of a function over all positive quantum operators bounded above by the identity operator. That is:

$$C_\Phi(\hat{\rho}) = \sup_{E:\ 0\leqslant E\leqslant \mathbb{1}} [\mathrm{tr}(\hat{\rho} - \Phi(\hat{\rho}))E]$$

We can then proceed in the following way:

$$\begin{aligned}C_\Phi(\sum_i P_i\ \hat{\rho}\ P_i) &= \sup_{E:\ 0\leqslant E\leqslant \mathbb{1}} \left[\mathrm{tr}\left(\sum_i P_i\ \hat{\rho}\ P_i - \Phi(\sum_i P_i\ \hat{\rho}\ P_i)\right) E\right]\\ &= \sup_{E:\ 0\leqslant E\leqslant \mathbb{1}} \left[\mathrm{tr}\left(\sum_i (P_i\ \hat{\rho}\ P_i - P_i\ \Phi(\hat{\rho})\ P_i)\right) E\right]\\ &= \sum_i \sup_{E:\ 0\leqslant E\leqslant \mathbb{1}} \left[\mathrm{tr}\left((P_i\ \hat{\rho}\ P_i - P_i\ \Phi(\hat{\rho})\ P_i)\right) E\right]\\ &= \sum_i C_\Phi(P_i\ \hat{\rho}\ P_i)\end{aligned}$$

Due to the linearity and additivity of the trace [54], we may take the summation sign outside of the trace function. This can be seen in the third equality of the proof.

(F5b): Recall that this property involves the rank-1 projector P_u, such that $P_u = |u\rangle\langle u|$. We progress by a simple calculation of $C_\Phi(\hat{\rho} \otimes P_u)$.

$$\begin{aligned} C_\Phi(\hat{\rho} \otimes P_u) &= \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} \otimes P_u - \Phi(\hat{\rho} \otimes P_u)|\right) \\ &= \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} \otimes |u\rangle\langle u| - \Phi(\hat{\rho} \otimes |u\rangle\langle u|)|\right) \\ &= \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} \otimes |u\rangle\langle u| - \Phi(\hat{\rho} \otimes |u\rangle\langle u|)|\right) \\ &= \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} \otimes |u\rangle\langle u| - \Phi(\hat{\rho}) \otimes |u\rangle\langle u||\right) \\ &= \frac{1}{2}\mathrm{tr}\left(|[\hat{\rho} - \Phi(\hat{\rho})] \otimes |u\rangle\langle u||\right) \\ &= \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} - \Phi(\hat{\rho})|\right) \cdot \mathrm{tr}\left(||u\rangle\langle u||\right) \\ [\mathrm{tr}\left(||u\rangle\langle u||\right) = 1] \Rightarrow \quad &= \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} - \Phi(\hat{\rho})|\right) \; = \; C_\Phi(\hat{\rho}) \end{aligned}$$

Hence, the dephasing distance measure of coherence is indeed invariant under a tensor product with a rank-1 projector.

This tells us that the (original) functional $C_\Phi(\hat{\rho})$ is a coherence measure. We can state that it is a good coherence measure from the fact that $\Phi(\hat{\rho})$ maps to the closest incoherent state to $\hat{\rho}$. It is also worth noting that the dephasing map exists in infinite-dimensional Hilbert spaces. That is, we may introduce $\Phi_\infty(\hat{\rho})$, defined to be the following operation.

$$\Phi_\infty(\hat{\rho}) = \sum_{n=0}^{\infty} |n\rangle\langle n|\hat{\rho}|n\rangle\langle n|$$

Hence, we can suggest that $C_\Phi(\hat{\rho})$ can exist in infinite-dimensional systems. We shall now discuss a potential coherence measure that definitely does not hold in infinite-dimensional Hilbert Spaces.

4.6. **Distance from the Maximally Coherent State.** Nearly all coherence measures to be found in the literature are distance measures that measure from an incoherent state to a coherent state. In order to alternatively quantify the coherence of a given quantum state, we may be able to measure its distance from a maximally coherent state, if indeed such a state exists. In fact, we have a standard definition for the maximally coherent state in d-dimensional Hilbert spaces [12]:

$$|\varphi_d\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} |n\rangle$$

This is defined to be the state in which any d-dimensional state can be prepared, by the use of incoherent operations. It is obvious to see that this state is not defined in infinite-dimensional systems. We may form a density matrix of this state, by performing the outer product of $|\varphi_d\rangle$.

$$|\varphi_d\rangle\langle\varphi_d| = \frac{1}{d}\left(\sum_{n=0}^{d-1}|n\rangle\right)\cdot\left(\sum_{m=0}^{d-1}\langle m|\right) = \frac{1}{d}\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 1\end{pmatrix}$$

The distance measure (induced by the trace norm), involving this maximally coherent density matrix, may be a measure of coherence. We define it concretely below.

Definition 4.9. Given a quantum state $\hat{\rho}$, the Distance from Coherence is defined in the subsequent way.

$$C_d(\hat{\rho}) = \frac{1}{2}\mathrm{tr}\left(|\hat{\rho} - |\varphi_d\rangle\langle\varphi_d||\right)$$

where $|\varphi_d\rangle\langle\varphi_d|$ is defined above.

As promising as this coherence measure looks, a simple example belies any validity of $C_d(\hat{\rho})$ being a coherence measure.

(C1): Let $\hat{\delta} = |1\rangle\langle 1|$ such that $\hat{\delta} \in I$. Then we may compute $C_d(\hat{\delta})$ in the following way.

$$C_d(\hat{\delta}) = \frac{1}{2}\mathrm{tr}\left(||1\rangle\langle 1| - |\varphi_d\rangle\langle\varphi_d||\right)$$

$$= \frac{1}{2}\mathrm{tr}\left(\left|\begin{pmatrix}0 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0\end{pmatrix} - \frac{1}{d}\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 1\end{pmatrix}\right|\right)$$

$$= \frac{1}{2}\mathrm{tr}\left(\left| - \frac{1}{d}\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ 1 & (1-d) & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 1\end{pmatrix}\right|\right)$$

This, of course, is not equal to zero because the trace of the resulting matrix is non-zero. Therefore, $C_d(\hat{\delta}) \neq 0$ and we can state that this functional violates (C1).

In order to remedy this breach of (C1), I thought about a new functional, $\tilde{C}_d(\hat{\rho})$:

$$\tilde{C}_d(\hat{\rho}) = 1 \ - \ C_d(\hat{\rho})$$

However, the example mentioned above violates this amendment also. It could also be shown that $\tilde{C}_d(\hat{\rho})$ violates the convexity property that a coherence measure must possess. It is also important to recall that the maximally coherent state, $|\varphi_d\rangle$, is not defined in infinite-dimensional systems. Thus, any functional involving such a state would not be a suitable coherence measure in infinite-dimensional Hilbert spaces.

It is known that some coherence measures, that are well-defined in finite-dimensions, become unstable when extended to infinite-dimensional systems. Below, I will discuss the stability of two well-renowned, previously discussed, coherence measures in infinite dimensions.

5. Measuring the Coherence of Canonical Coherent States

We must refer back to the canonical coherent states defined in the introduction of this paper. Roy Glauber discovered these states back in 1963 [9] and in this section, we attempt to measure the coherence of these states. The fascination behind this idea revolves around the fact that Glauber defined these states as Coherent States. Now that the theory of quantum coherence has vastly improved since the early 60's, these states have since been renamed as the canonical coherent states. The surrounding ambiguity of quantum coherence begs the question: How coherent are the canonical coherent states defined by Roy Glauber in 1963?

Recall from Properties 1.1 - 1.3, Glauber's canonical coherent states are defined in infinite-dimensional Hilbert spaces. Thus, we need a coherence measure that is well-defined in infinite-dimensional Hilbert spaces. In this section, we shall discuss a property that a coherence measure must satisfy in order for this to be the case. But first, we must discuss some background information, as we did for the finite-dimensional case, before we can begin to discuss infinite-dimensional coherence measures.

5.1. **Incoherent States and Operations in Infinite Dimensions.** It is important to extend the definitions given in this manuscript to the infinite-dimensional Hilbert space spanned by the number basis. This is so we can paint a full picture of the system, providing intuition for the extra property that a coherence measure must satisfy in order to be defined in infinite dimensions. We shall define incoherent states and incoherent operations in infinite dimensions.

Definition 5.1. An ∞-dimensional incoherent state, $\hat{\delta} \in I$, is defined to be a diagonal density matrix with respect to a chosen basis. That is to say, an incoherent state defined in the number basis is given by the following equation.

$$\hat{\delta} = \sum_{i=0}^{\infty} \delta_i |i\rangle\langle i|$$

The δ_i satisfy both $0 \leqslant \delta_i \leqslant 1$ and $\sum_{i=0}^{\infty} \delta_i = 1$.

As the number basis spans an infinite-dimensional Hilbert space, we can merely extend our definition of finite-dimensional incoherent states. We shall now give the definition for incoherent operations that allow for sub-selective measurements, again, in terms of Kraus operators.

Definition 5.2. An infinite-dimensional incoherent measuring operation, $\Phi_\infty(\hat{\rho})$, is given by a set of Kraus operators, $\{K_n\}_{0\leqslant n\leqslant\infty}$, satisfying both $K_n I K_n^\dagger \subset I$ and $\sum_n^\infty K_n^\dagger K_n = \mathbb{1}$. Each Kraus operator, K_n, is of dimension $d_n \times \infty$ for each n. Thus, we can say that an infinite-dimensional incoherent measuring operation is given in the following form.

$$\Phi_\infty(\hat{\rho}) = \sum_n^{\infty} K_n \hat{\rho} K_n^\dagger$$

This is the infinite-dimensional extension of the operations involved in the (C2b) property. I stated these generalised definitions for the infinite-dimensional case, so that it can be easily shown that the properties (C1)-(C3) hold for coherence measures in infinite dimensions. As mentioned previously, there is an extra property that coherence measures have to satisfy in order to be defined in infinite dimensions. I shall tweak the definition of (C4) given by Zhang et. al. [55].

(C4): Let $C(\hat{\rho})$ be a coherence measure that satisfies the properties (C1) - (C3) in finite dimensions. Then, we say that $C(\hat{\rho})$ is defined in infinite-dimensional systems if $C(\hat{\rho}) < \infty$ for a fixed average particle number, $\langle n\rangle$.

This condition may seem abstract if one is not familiar with mathematical analysis. Put into words, we require that an infinite-dimensional coherence measure does not diverge for a fixed particle number $\langle n\rangle$. Though we have not discussed particle numbers in this manuscript, we can see that the average particle number of a system described by the canonical coherent state can be written in familiar terms.

We may calculate the average number of particles residing in a system described by the canonical coherent state using the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$. Recall that the expectation value of an operator $\hat{X}$ in a pure quantum state $|\varphi\rangle$ is given by:

$$\langle \hat{X}\rangle_\varphi = \langle\varphi|\hat{X}|\varphi\rangle$$

Knowing that the canonical coherent state is an eigenstate of the annihilation operator, we may compute $\langle n \rangle_\alpha$ as the expectation value of the number operator [56]. That is:

$$\langle n \rangle_\alpha = \langle \hat{N} \rangle_\alpha = \langle \alpha | \hat{N} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\hat{a}|\alpha\rangle|^2 = ||\alpha| e^{i\theta}|^2$$

$$\therefore \quad \langle n \rangle_\alpha = |\alpha|^2$$

It is interesting to note that the average number of particles residing in a system described by a canonical coherent state is related to the amplitude, $|\alpha|$, of the system. Thinking about Schrödinger's wave mechanics, this correlation could be due to constructive interference of the individual waves describing each particle. The more particles that reside in a particularly coherent system, the greater the chance of constructive interference occurring between each particle. Thus, an increase in amplitude of the entire system occurs as a result. It would be interesting to see an investigation done into the effects of constructive interference of a coherent system: A possible pathway of discussion, perhaps.

The above argument suggests that we can investigate (C4), for a canonically coherent system, by first fixing $|\alpha|^2$ and then testing whether a coherence measure converges. This line of reasoning can be made clearer by an example.

5.2. **The Relative Entropy of the Canonical Coherent State.** The relative entropy is a coherence measure that seems to work in infinite dimensions. We may write the relative entropy in terms of probability distributions. Known as the Shannon entropy [57], for a set of probability distributions $\{Q_i\}$, it can be defined in the following way:

$$C_{\text{Rel Ent.}}(\hat{\rho}) = - \sum_i Q_i \cdot \log(Q_i)$$

It turns out that there is a simple way of calculating the probability distribution of detecting n particles residing in a canonically coherent system. The calculation is as follows.

The canonical coherent state can be given by $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$. Therefore, we can use this to compute the probability distribution [58], $Q_{\text{CS}}(n) = |\langle n|\alpha\rangle|^2$.

$$\begin{aligned}
\langle n|\alpha\rangle &= \langle n| \cdot e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \langle n|m\rangle \\
&= e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \qquad \text{since } \langle n|m\rangle = 1 \Leftrightarrow m = n \\
\Rightarrow \qquad & |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{\alpha^{2n}}{n!}
\end{aligned}$$

Remember that we defined the average number of particles in the canonically coherent system, $\langle n\rangle_\alpha$, to be $\langle n\rangle_\alpha = |\alpha|^2$. Thus,

$$Q_{\text{CS}}(n) = |\langle n|\alpha\rangle|^2 = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}$$

This is, in fact, a Poisson distribution, showing the probability of a given number of particles being detected in a fixed time frame, independently of the last observation. We may use this distribution in the definition of $C_{\text{Rel Ent.}}(\hat{\rho})$ proposed above.

$$\begin{aligned}
C_{\text{Rel Ent.}}(\hat{\rho}) &= -\sum_{n=0}^{\infty} Q_{\text{CS}}(n)\cdot\log\left(Q_{\text{CS}}(n)\right)\\
&= -\sum_{n=0}^{\infty} e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}\cdot\log\left(e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}\right)\\
&= e^{-\langle n\rangle}\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{n!}\cdot\log\left(\frac{n!}{\langle n\rangle^n e^{-\langle n\rangle}}\right)\\
&= e^{-\langle n\rangle}\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{n!}\cdot\left[\log(n) - n\log(\langle n\rangle) + \langle n\rangle\log(e)\right]\\
&= e^{-\langle n\rangle}\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{n!}\cdot\left[\log(n) + \langle n\rangle\log(e)\right] - \sum_{m=0}^{\infty}\frac{\langle n\rangle^m}{m!}m\log(\langle n\rangle)\\
&= e^{-\langle n\rangle}\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{n!}\cdot\left[\log(n) + \langle n\rangle\log(e)\right] - \sum_{m=0}^{\infty}\frac{\langle n\rangle^{m-1}}{(m-1)!}\frac{\langle n\rangle}{m}m\log(\langle n\rangle)
\end{aligned}$$

After relabelling the indices of the last summation and simplifying its summand, we get the following result.

$$\begin{aligned}
C_{\text{Rel Ent.}}(|\alpha\rangle) &= e^{-\langle n\rangle}\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{n!}\cdot\log(n) + \langle n\rangle\log(e) - \langle n\rangle\log(\langle n\rangle)\\
&= e^{-\langle n\rangle}\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{n!}\cdot\log(n) - \langle n\rangle\log\left(\frac{\langle n\rangle}{e}\right)
\end{aligned}$$

By the property of (C4), we wish for this series to converge for fixed $\langle n\rangle$. We can observe convergence via the ratio test.

Theorem 5.3. Let $\sum_{n=1}^{\infty} a_n$ be a series. We say that this series converges absolutely if and only if the following inequality is true.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

This is known as the ratio test.

We may use Theorem 5.3 to test whether $C_{\text{Rel Ent.}}(|\alpha\rangle)$ converges. In order to prepare the series for the ratio test, we may write each term in the summand as $a_n = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}\log(n)$. As $\langle n\rangle$ is fixed, we may call it x in the following argument, so as to avoid confusion. We start by calculating the input of Theorem 5.3.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{e^{-x}\frac{x^{(n+1)}}{(n+1)!}\log((n+1)!)}{e^{-x}\frac{x^n}{n!}\log(n!)}\right| = \left|\frac{x^{(n+1)}n!\log((n+1)!)}{x^n(n+1)!\log(n!)}\right|$$

$$= \left|\frac{x\cdot n!\log((n+1)n!)}{n!(n+1)\log(n!)}\right| = \left|\frac{x\log((n+1)n!)}{(n+1)\log(n!)}\right|$$

$$= \left|\frac{x\log(n+1)+x\log(n!)}{(n+1)\log(n!)}\right| = \left|\frac{x\log(n+1)}{(n+1)\log(n!)} + \frac{x}{(n+1)}\right|$$

We can then apply the limit to this input. One can see what happens as n gets large.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x\log(n+1)}{(n+1)\log(n!)} + \frac{x}{(n+1)}\right| = 0$$

$$\text{because } (n+1)\log(n!) >> \log(n+1) \text{ as } n\to\infty$$

Therefore, the series converges absolutely with an infinite radius of convergence. Thus, the relative entropy is suitable for measuring the coherence of canonical coherent states residing in any infinite-dimensional system. I used Mathematica to plot the coherence as measured by relative entropy against the average number of particles residing in the system.

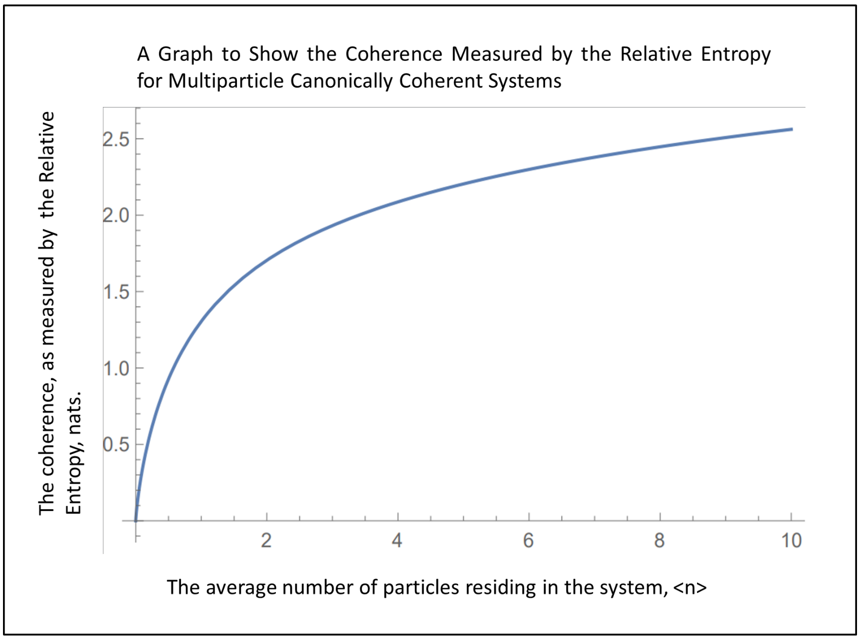

FIGURE 1. The coherence of the canonically coherent system, retaining a various number of particles.

From Figure 1, we see that the coherence tends to a finite value as $\langle n \rangle \to \infty$. We can extrapolate the data collected and find an approximate value for the limit.

$$\lim_{\langle n \rangle \to \infty} C_{\text{Rel Ent.}}(|\alpha\rangle) \approx 2.7 \text{ nats}$$

Though it may seem rather abritrary, this is an answer to the main question posed by this manuscript. The coherence of relative entropy is given in nats: the natural unit of information [59]. Interesting future work could include calculating the maximum value that could be attained by the relative entropy. This would provide a reference point so that we can compare this approximation to both the minimum and maximum value that relative entropy could achieve. We do know, however, that the minimum value that relative entropy reaches is zero. Hence, at the very least, we can say that the canonical coherent state is far from incoherent as the number of particles residing in the system gets very large. It is important to note the point at which $\langle n \rangle = 0$: The zero particle system is perfectly incoherent. This can be explained mathematically: The vacuum state $|0\rangle$ describes the system that has no particles residing in it also the density matrix for this system, $|0\rangle\langle 0|$, is diagonal and is therefore representing an incoherent system.

In order to conclude this investigation into the relative entropy, we shall discuss the preliminary work I carried out before researching this line of thought. I used the relative entropy to measure the coherence of the two-dimensional canonical coherent state.

By the interesting work of Miranowicz et. al. [60], a numerical method was shown to reduce the canonical coherent state into finite-dimensional states. The two-dimensional canonical coherent state, whose coherence of which we shall measure, is given below.

$$|\alpha\rangle_{(1)} = \cos|\alpha||0\rangle + e^{i\phi_0} \sin|\alpha||1\rangle$$

In the definition given by Miranowicz et. al., a fixed phase ϕ_0 and a fluctuating amplitude $|\alpha|$ are considered. We wish to plot the coherence as the amplitude varies. In order to do this, we must rewrite the coherence measure. The relative entropy may be calculated in terms of, $\{\lambda_i\}$, the set of eigenvalues of the input state.

$$C_{\text{Rel Ent.}}(\hat{\rho}) = -\sum_i \lambda_i \cdot \log\left(\frac{1}{\lambda_i}\right)$$

We can see that the eigenvalues of $|\alpha\rangle_{(1)}$ are easy to find once we apply the outer product using its Hermitian conjugate:

$$|\alpha\rangle\langle\alpha|_{(1,1)} = (\cos|\alpha|)^2|0\rangle\langle 0| + e^{i\phi_0}\cos|\alpha|\sin|\alpha||1\rangle\langle 0| + e^{-i\phi_0}\cos|\alpha|\sin|\alpha||0\rangle\langle 1| + (\sin|\alpha|)^2|1\rangle\langle 1|$$

$$\widehat{=} \begin{pmatrix} (\cos|\alpha|)^2 & e^{i\phi_0}\cos|\alpha|\sin|\alpha| \\ e^{-i\phi_0}\cos|\alpha|\sin|\alpha| & (\sin|\alpha|)^2 \end{pmatrix}$$

It is clear to see that the eigenvalues of this matrix are dependent on the amplitude of the system. Finding the eigenvalues of the matrix, I then substituted them into our new definition for the relative entropy and used Mathematica to plot the results. Figure 2, in which the amplitude of the system is denoted by $|\alpha|$, can be seen below.

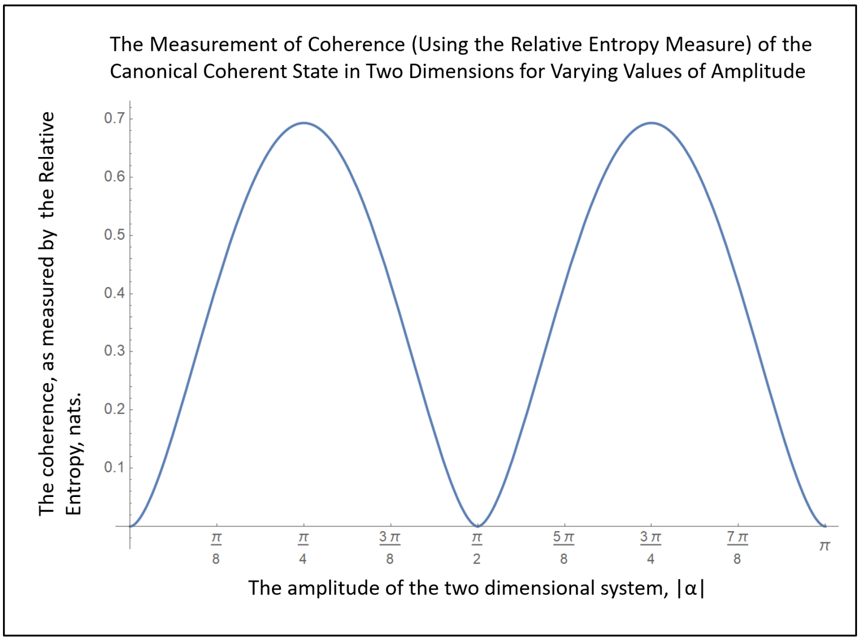

FIGURE 2. The coherence of the canonically coherent two-dimensional system for varying values of amplitude, $|\alpha|$.

As a result of the eigenvalues being dependent on $|\alpha|$, it appears that the coherence is dependent on the amplitude of the system too. We see the same zero coherence phenomenon for the point of zero amplitude. Referring back to the definition of $|\alpha\rangle_{(1)}$, we see that the state with zero amplitude is indeed the vacuum state, $|0\rangle$. One obvious conclusion we can draw is that the vacuum state is incoherent in the number basis. The periodicity of the coherence with respect to the amplitude is yet to be explained, however. This may be due to the loss of information we incur as a result of reducing the canonical coherent state to two dimensions.

We shall now discuss a coherence measure that fails upon application to an infinite-dimensional system. That is, it fails (C4).

5.3. **The l_1 Norm of the Canonical Coherent State.** Recall from Definition 4.5, we defined the l_1 norm of coherence as the sum of all the off-diagonal elements. In order to see how this applies to the canonical coherent state, we must construct the canonically coherent density matrix, $|\alpha\rangle\langle\alpha|$. We shall use the same outer product method that has been displayed throughout this section.

$$|\alpha\rangle\langle\alpha| = e^{-|\alpha|^2}\left[\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\right]\left[\sum_{j=0}^{\infty}\frac{(\bar{\alpha})^j}{\sqrt{j!}}\langle j|\right]$$

$$\widehat{=}\ e^{-|\alpha|^2}\begin{pmatrix} 1 & \bar{\alpha} & \frac{\bar{\alpha}^2}{\sqrt{2}} & \cdots & \frac{\bar{\alpha}^n}{\sqrt{n!}}\cdots \\ \alpha & |\alpha|^2 & \cdots & \cdots & \frac{\alpha\bar{\alpha}^n}{\sqrt{n!}}\cdots \\ \frac{\alpha^2}{\sqrt{2}} & \vdots & \frac{|\alpha|^4}{2} & \cdots & \frac{\alpha^2\bar{\alpha}^n}{\sqrt{2}\sqrt{n!}}\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha^n}{\sqrt{n!}} & \frac{\bar{\alpha}\alpha^n}{\sqrt{n!}} & \frac{\bar{\alpha}^2\alpha^n}{\sqrt{2}\sqrt{n!}} & \cdots & \frac{\alpha^n\bar{\alpha}^n}{n!}\cdots \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

Recall, the property (C4) requests that we fix $\langle n\rangle_\alpha = |\alpha|^2$. From this, we can see rather straightforwardly that adding all of the (strictly positive) elements of an infinite-dimensional square matrix will not converge and in fact diverge to infinity. That is to say, we require the following double series to converge.

$$C_{l_1}(|\alpha\rangle\langle\alpha|) = e^{-\langle n\rangle}\sum_{n\neq j}^{\infty}\left|\frac{\alpha^n}{\sqrt{n!}}\cdot\frac{(\bar{\alpha})^j}{\sqrt{j!}}\right|$$

We can prove this much more robustly using the theory of partial sums. The following theorem, from the work of Habil [61], helps us to understand this divergence idea for double series more clearly.

Theorem 5.4. A double series of non-negative terms $\sum_{n,j=0}^{\infty} z(n,j)$ converges if and only if the set of partial sums $\{s(n,j) : n,j\in\mathbb{N}\}$ is bounded.

We can prove non-convergence of a double series by showing that the set of partial sums is unbounded. We can proceed by proof of contradiction.

We may assume that there exists a supremum, $A\in\mathbb{R}$, such that the set of partial sums of the series we wish to investigate, S, is bounded. That is,

$$S = \left\{e^{-\langle n\rangle}\sum_{a=1}^{n}\sum_{b=1}^{j}\frac{\alpha^a}{\sqrt{a!}}\cdot\frac{(\bar{\alpha})^b}{\sqrt{b!}}\cdot(1-\delta_{a,b})\ :\ n,j\in\mathbb{N}\right\}\leqslant A$$

Consider the case where $n = 2$ and $j = 2$. We can calculate the explicit equation for the partial sum, $S_{2,2}$, bounded by the supremum A.

$$S_{2,2} = e^{-\langle n\rangle} \sum_{a=1}^{2}\sum_{b=1}^{2} \frac{\alpha^a}{\sqrt{a!}} \cdot \frac{(\bar{\alpha})^b}{\sqrt{b!}} \cdot (1-\delta_{a,b})$$

$$= e^{-\langle n\rangle}\left(\alpha + \bar{\alpha} + \frac{\alpha^2}{\sqrt{2}} + \frac{\bar{\alpha}^2}{\sqrt{2}} + \frac{\langle n\rangle\,\alpha}{\sqrt{2}} + \frac{\langle n\rangle\,\bar{\alpha}}{\sqrt{2}}\right) \leqslant A$$

Here, we have used the equation $|\alpha|^2 = \alpha \cdot \bar{\alpha} = \langle n\rangle$. If we consider the case where $n = 3$ and $j = 3$, we achieve the following expression for $S_{3,3}$, noting that A is no longer the supremum of the set.

$$S_{3,3} = e^{-\langle n\rangle} \sum_{a=1}^{3}\sum_{b=1}^{3} \frac{\alpha^a}{\sqrt{a!}} \cdot \frac{(\bar{\alpha})^b}{\sqrt{b!}} \cdot (1-\delta_{a,b})$$

$$= e^{-\langle n\rangle}\Bigg(\alpha + \bar{\alpha} + \frac{\alpha^2}{\sqrt{2}} + \frac{\bar{\alpha}^2}{\sqrt{2}} + \frac{\langle n\rangle\,\alpha}{\sqrt{2}} + \frac{\langle n\rangle\,\bar{\alpha}}{\sqrt{2}}$$

$$+ \frac{\alpha^3}{\sqrt{6}} + \frac{\alpha^3\bar{\alpha}}{\sqrt{6}} + \frac{\bar{\alpha}^2\alpha^3}{\sqrt{6}} + \frac{\alpha^2\bar{\alpha}^3}{\sqrt{6}} + \frac{\alpha\bar{\alpha}^3}{\sqrt{6}} + \frac{\bar{\alpha}^3}{\sqrt{6}}\Bigg)$$

$$= A + e^{-\langle n\rangle}\left(\frac{\alpha^3}{\sqrt{6}} + \frac{\alpha^3\bar{\alpha}}{\sqrt{6}} + \frac{\bar{\alpha}^2\alpha^3}{\sqrt{6}} + \frac{\alpha^2\bar{\alpha}^3}{\sqrt{6}} + \frac{\alpha\bar{\alpha}^3}{\sqrt{6}} + \frac{\bar{\alpha}^3}{\sqrt{6}}\right) \nleqslant A$$

This tells us that, if we are given $n, j \in \mathbb{N}$, we may always find $n+1, j+1 \in \mathbb{N}$ such that $s(n+1, j+1) > s(n, j)$. Thus, there is no supremum for the set of partial sums, $\{s(n,j) : n, j \in \mathbb{N}\}$, of our double series. Hence, the double series we are investigating is non-convergent and therefore $C_{l_1}(|\alpha\rangle\langle\alpha|)$ violates (C4). We can now state that the l_1 norm of coherence would not be a suitable measure over the canonically coherent system.

6. Conclusion

In this manuscript, we defined incoherent states and incoherent operations that are engineered in the number basis. We have stated and explained the four tenets that functionals must satisfy in order to be called coherence measures in finite-dimensional systems. Using these, we proved these tenets for two widely-renowned coherence measures and three promising candidates.

The relative entropy and l_1 norm of coherence are widely known and proven coherence measures. In this paper, we provided an original proof that ratifies the claim that the l_1 norm of coherence is indeed a coherence measure. The proof clearly takes advantage of the well-documented relationship between coherence and entanglement measures, of which we have not explicitly discussed here. This may be a possible avenue of investigation.

Unfortunately, the very promising coherence measure induced by fidelity did not satisfy (C2b). I gave an example of a quantum state, and a set of Kraus operators, for which the fidelity of quantum coherence increased on average under sub-selective measurements of the quantum state. Thus, this paper provided an original testimony for the case against the fidelity of coherence being a coherence measure.

The three original candidates I proposed in this paper provided insight on how to approach the construction of coherence measures. The first distance measure I constructed utilised the dephasing map. Knowing that the dephasing map diagonalises any density matrix, I postulated that the distance from the original state to its diagonalised form would provide a measure of coherence. Indeed, we saw that this original measure satisfied the four tenets stated in this paper.

Another original distance measure we discussed involved the commutator, defined as $C_{[\cdot,\cdot]}(\hat{\rho}) = \left\|[\hat{\rho}, \hat{N}]\right\|_1$. The intuition behind this functional was to treat the absolute value of the commutator as a measure of distance. The resulting value would tell us how close a state $\hat{\rho}$ would be to the diagonal matrix $\hat{N}$. An in-depth look into the (C2b) property soon belied any validity $C_{[\cdot,\cdot]}(\hat{\rho})$ had of being a coherence measure. It was difficult to show that it decreased under a partial trace operation due to complicated Lie algebras at play.

The third candidate we proposed involved the maximally coherent state. Having been understood for years, the maximally coherent state is said to be the state that can be prepared into any d-dimensional state with the use of incoherent operations. By computing the trace distance between the maximally coherent state and an input state $\hat{\rho}$, we attempted to find a different form of measure. Instead, this possible measure would quantify how distant the input state would be from the maximally coherent state. A straightforward example debased the claim that $C_d(\hat{\rho})$ could be a coherence measure. It is also important to note that this measure would not be valid in infinite dimensions due to the fact that the maximally coherent state is not defined in infinite dimensions.

For the stated functionals that deserve the right to be called coherence measures, we stated an extra property that they needed to satisfy in order to function as coherence measures in infinite-dimensional systems. We saw that relative entropy satisfied this new property, allowing us to measure the coherence of the canonical coherent state. I stated that the l_1 norm of coherence fails upon application to an infinite-dimensional system, proving that the functional is unstable for a fixed average particle number. Though we have seen an original approach in this paper, the claim that the l_1 norm of coherence fails in an infinite-dimensional setting is nothing new. I implemented this to build upon the already insurmountable case for how difficult it is to construct coherence measures. To have only the relative entropy of coherence, out of the five potential coherence measures proposed in this paper, to be valid in infinite dimensions is nothing short of astounding in my estimation.

Throughout this manuscript, I have clearly considered other avenues of investigation, particularly when constructing coherence measures and the tools required to do so. I hope that this plants the seed for yet more coherence measures to come to fruition.

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